Full Waveform Inversion via Matched Source Extension

Guanghui Huang and William W. Symes

TRIP, Department of Computational and Applied Mathematics

May 1, 2015, TRIP Annual Meeting

Outline

Outline

Overview

Assume that the received pressure field $p(\mathbf{x}_r, t; \mathbf{x}_s)$ generated by a causal isotropic point radiator at source position $\mathbf{x} = \mathbf{x}_s$ satisfies the constant density acoustic wave equation,

$$\frac{1}{v^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \delta(\mathbf{x} - \mathbf{x}_s) f(t) \quad \text{in } \mathbb{R}^2. \tag{1}$$

$$p|_{t=0} = \left.\frac{\partial p}{\partial t}\right|_{t=0} = 0 \tag{2}$$

Let's introduce the forward modeling operator $S[v]$ to relate the velocity $v(x, z)$ and wavelet function $f(t)$ to the scattering field at the receiver $\mathbf{x}_r$,

$$S[v, f](\mathbf{x}_r, t; \mathbf{x}_s) = p(\mathbf{x}_r, t; \mathbf{x}_s). \tag{3}$$

Full Waveform Inversion

Given recorded traces $d(\mathbf{x}_r, t; \mathbf{x}_s)$, find velocity v and wavelet function f such that $S[v, f] = d$.

- FWI via data fitting,

$$J_{\mathrm{FWI}}[v, f] = \frac{1}{2} \sum_{\mathbf{x}_r, \mathbf{x}_s} \int |S[v, f](\mathbf{x}_r, t; \mathbf{x}_s) - d(\mathbf{x}_r, t; \mathbf{x}_s)|^2 dt. \quad (4)$$

- FWI objective function is quadratic with respect to f, but highly nonlinear and nonconvex in velocity v (frequency dependent).
- Cycle skipping problem (eg. Symes, 1994).

Outline

Extended Modeling & Null Space

Extended Modeling:

Let $\bar{f}(\mathbf{x}_r, \mathbf{x}_s, t)$ be the extended model of $f(t)$, define the extended modeling operator

$$\bar{S}[v, \bar{f}] = \bar{p}(\mathbf{x}_r, t; \mathbf{x}_s)$$

where $\bar{p}$ is the solution of (1)-(2) with source function being $\bar{f}$.

Null Space (Annihilator):

- Differential semblance operator $A = \partial_{z_s}$ (see Symes (1994));
- t-moment operator after source signature deconvolution $A = tf^{-1}(t)$ (eg. deconvolution-based by Luo and Sava (2011), AWI by Warner (2014)).

MSWI

Matched Source Waveform Inversion (MSWI) is stated as follows,

$$J_{\mathrm{MS}}[v] = \frac{1}{2}\sum_{\mathbf{x}_r,\mathbf{x}_s}\int |A\bar{f}|^2 dt \tag{5}$$

$$s.t.\ \bar{S}[v,\bar{f}](\mathbf{x}_r,t;\mathbf{x}_s) = d(\mathbf{x}_r,t;\mathbf{x}_s). \tag{6}$$

Key feature: Even given wrong velocity, data fitting is perfect, hence no cycle skipping problem!

Outline

Factorization of Tangent Operator

Under single arrival approximation, we have

$$\bar{S}[v, \bar{f}](\mathbf{x}_r, t; \mathbf{x}_s) \approx a(\mathbf{x}_r, \mathbf{x}_s)\bar{f}(\mathbf{x}_r, \mathbf{x}_s, t - \tau(\mathbf{x}_r, \mathbf{x}_s))$$

Then taking the first order variation of $\bar{S}[v, \bar{f}]$ formally gives us the desired factorization of operator

$$(D\bar{S}[v]\delta v)\bar{f}(\mathbf{x}_r, t; \mathbf{x}_s) \approx a(\mathbf{x}_r, \mathbf{x}_s)\frac{\partial}{\partial t}\bar{f}\left(\mathbf{x}_r, \mathbf{x}_s, t - \tau(\mathbf{x}_r, \mathbf{x}_s)\right)(-D\tau[v]\delta v)$$
$$\triangleq \bar{S}[v, Q[v, \delta v]\bar{f}].$$

where $D\tau[v]$ is the tangent operator of traveltime function, and $Q[v, \delta v]\bar{f}$ is bilinear operator with respect to δv and f with

$$Q[v, \delta v] = -(D\tau[v]\delta v)\frac{\partial}{\partial t}.$$

Backprojection of Traveltime Differences

Taking the first order perturbation of J_{MS}, we have

$$DJ_{\mathrm{MS}}[v]\delta v \approx \sum_{\mathbf{x}_r,\mathbf{x}_s} \int A^T A\bar{f}(D\tau[v]\delta v \bar{f}_t)dt$$

Hence the gradient is given by,

$$g \approx \sum_{\mathbf{x}_r,\mathbf{x}_s} D\tau[v]^T \left(\int (A^T A\bar{f})\bar{f}_t dt \right)$$

Here $D\tau[v]^T$ is the adjoint operator of $D\tau[v]$, which backprojects its arguments along rays.

Adjoint Source in the Gradient

Assume that data is noise-free and well approximated by geometric optics,

$$d(\mathbf{x}_r, t; \mathbf{x}_s) \approx a^*(\mathbf{x}_r, \mathbf{x}_s) f^*(t - \tau[v^*](\mathbf{x}_r, \mathbf{x}_s)). \tag{7}$$

Denote by $\Delta\tau(\mathbf{x}_r, \mathbf{x}_s) = \tau[v^*](\mathbf{x}_r, \mathbf{x}_s) - \tau[v](\mathbf{x}_r, \mathbf{x}_s)$, then

- For differential semblance operator $A = \partial_{z_s}$,

$$\int (A^T A\bar{f})\bar{f}_t dt \approx -\Big(\frac{a^*}{a}\Big)^2 \int \Big(\frac{\partial f^*}{\partial t}\Big)^2 dt (\frac{\partial}{\partial z_s})^T \Big(\frac{\partial}{\partial z_s} \Delta\tau[v](z_r, z_s)\Big).$$

- For t-moment operator, we have

$$\int (A^T A\bar{f})\bar{f}_t dt \approx -(a^*/a)^2 \Delta\tau[v](z_r, z_s).$$

Traveltime Tomography

Travel tomography attempts to minimize the difference between the computed traveltime and picked traveltime from data,

$$J_{\mathrm{TT}} = \frac{1}{2}\sum_{\mathbf{x}_r,\mathbf{x}_s} \|\tau[v](\mathbf{x}_r,\mathbf{x}_s) - \tau[v^*](\mathbf{x}_r,\mathbf{x}_s)\|^2,$$

The gradient of J_{TT} is given by,

$$\nabla J_{\mathrm{TT}} = -\sum_{\mathbf{x}_r,\mathbf{x}_s} D\tau[v]^T(\Delta\tau[v](z_r,z_s)).$$

Local Convexity of Hessian

The Hessian of J_{MS} at consistent data $A\bar{f} = 0$ is given by

$$D^2 J[v^*](\delta v, \delta v) \approx \sum_{\mathbf{x}_r, \mathbf{x}_s} \int |[A, Q](v^*, \delta v)\bar{f}|^2 dt$$

where $[A, Q] = AQ - QA$.

- for $A = \partial_{z_s}$,

$$D^2 J[v^*](\delta v, \delta v) \approx \int \left(\frac{\partial f^*}{\partial t}\right)^2 dt \left(\sum_{\mathbf{x}_r, \mathbf{x}_s} \left| \frac{\partial}{\partial z_s} D\tau[v^*]\delta v \right|^2 \right)$$

- for $A = tf^{-1}(t)$,

$$D^2 J[v^*](\delta v, \delta v) \approx \sum_{\mathbf{x}_r, \mathbf{x}_s} |D\tau[v^*]\delta v|^2$$

Hessian of J_{MS} is proportional to the Hessian of a traveltime objective function, and is as convex as tomographic objective.

Relation with DSO formulation

DSO formulation seeks the balance between data fitting and annihilator term, i.e.

$$J_\alpha[v,\bar{f}] = \frac{1}{2}\sum_{\mathbf{x}_r,\mathbf{x}_s}\int |\bar{S}[v,\bar{f}] - d|^2 dt + \frac{\alpha}{2}\sum_{\mathbf{x}_r,\mathbf{x}_s}\int |A\bar{f}|^2 dt.$$

- As $\alpha \to 0$, the gradient and Hessian of $\frac{1}{\alpha}J_\alpha$ is the same as J_{MS}.
- As $\alpha \to +\infty$, $A\bar{f} = 0$, then it's equivalent to minimize J_{FWI}

Outline

Example 1: Velocity Scan of J_{MS}

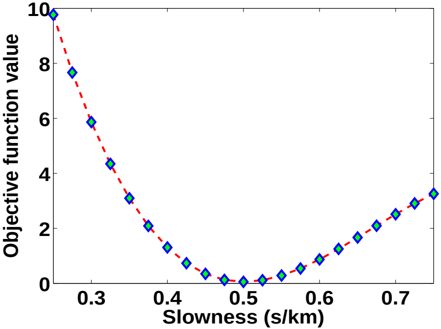


Figure: $J_{\mathrm{MS}}[v]$ for homogeneous velocity v, 0.25 s/km $\leq v^{-1} \leq$ 0.75 s/km. Correct velocity is 2 km/s.

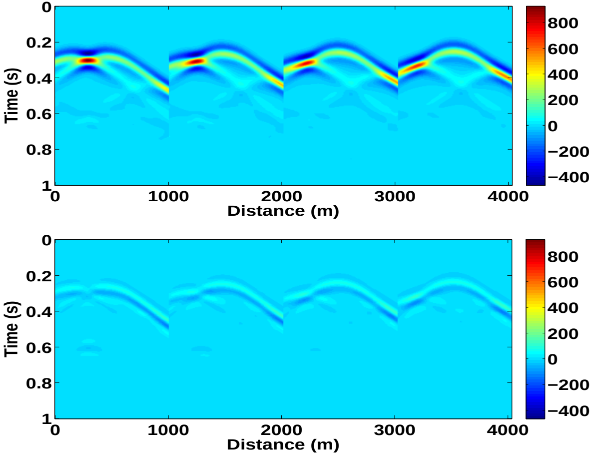
Example 2: Gaussian Lens

In this example, the target velocity consists of two Gaussian velocity anomalies embedded in a $v = 2$km/s background:

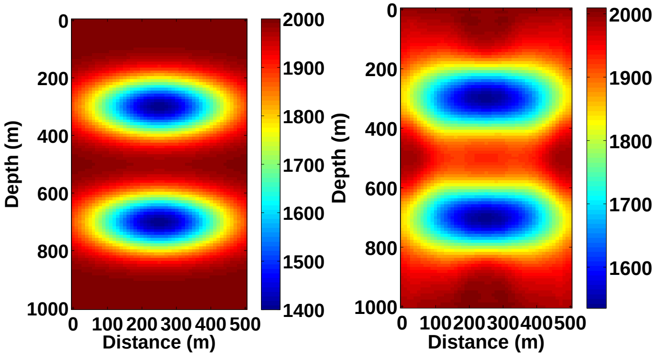
$$v(x,z) = 2 - 0.6e^{-\frac{(x-0.25)^2+(z-0.3)^2}{(0.2)^2}} - 0.6e^{-\frac{(x-0.25)^2+(z-0.7)^2}{(0.1)^2}},$$

where $x \in [0, 0.5]\mathrm{km}, z \in [0, 1]\mathrm{km}$. The initial model is given by the constant velocity $v_0 = 2$ km/s. Data is consisted of 50 shots and 99 receivers for each shot, which are uniformly distributed.

Example 2: Gaussian Lens



Example 2: Gaussian Lens



Example 2: Gaussian Lens

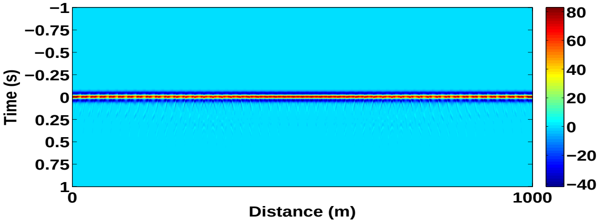


Figure: Extended source functions

Example 3: Big Gaussian

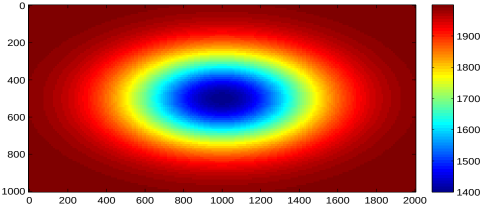


Figure: Low velocity Gaussian anomaly model with radius $250m \times 150m$ embedded in the constant background velocity $v_0 = 2000m/s$

Example 3: Big Gaussian

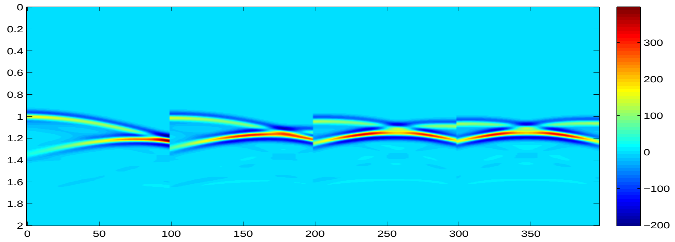


Figure: Receivers are uniformly distributed long $x_r = 1990$m from $z_r = 10$m to $x_r = 990$m for each shots, shots interval is 20m. The zero-phase ricker wavelet with main frequency $f = 10$Hz for generating the data.

Example 3: Big Gaussian

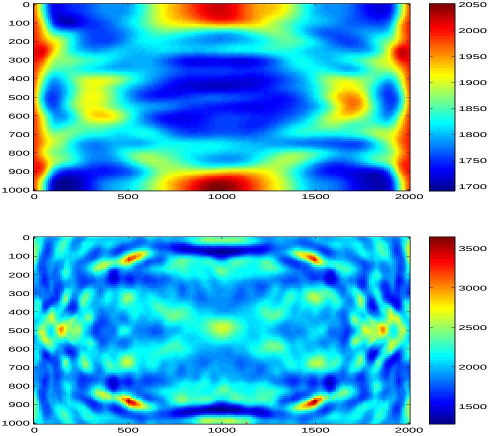


Figure: Inverted velocity after 50 iterations by MSWI (top) and FWI (bottom)

Outline

Conclusion and Discussion

Conclusion

- Establish the relation between waveform inversion and traveltime tomography;
- Convexity of objective function is independent of frequency content.
- MSWI still get stuck in local minima due to the strong multipaths.

Discussion

- Does perfect data fitting (no cycle skipping) mean avoiding local minima;
- How do we choose the extended model space and the related annihilator.

To be cont'd

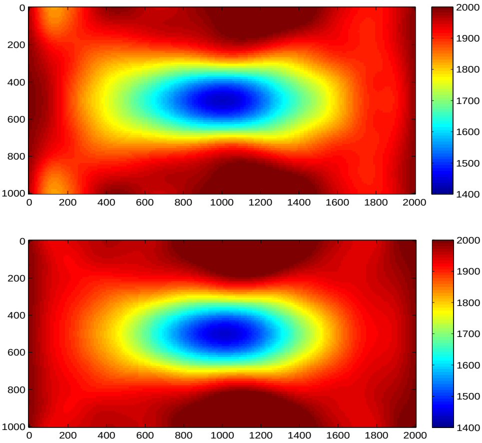


Figure: Inverted velocity after 100 iterations and 200 iterations

Thanks

- Total E&P USA, for funding me
- The Rice Inversion Project, for supporting me
- TRIP members, for helping me
- All of you, for listening to me