Subgrid modeling via mass lumping in constant density acoustics

William W. Symes, Igor S. Terentyev, Rice University*

SUMMARY

Finite difference simulations in discontinuous media are only first-order accurate regardless of the formal order of the method. Stairstep diffractions are a widely known manifestation of this first-order error. To recover the optimal convergence rate, we apply a finite element approach with mass lumping to a constant density acoustic wave equation. This approach results in an explicit second-order accurate difference scheme with specifically averaged sound velocity. Two numerical experiments confirm theoretical convergence rate. In particular, application of finite element discretizations with mass lumping leads to elimination of stairstep diffraction observed in simulations based on standard finite difference discretizations.

INTRODUCTION

Finite difference (FD) methods have long been accepted in a seismic industry as an easy-to-implement and computationally efficient tool for forward wave propagation simulations. these methods have been extensively studied (see, for example, Moczo et al. (2006) and references cited therein) and the properties of FD solutions in smooth media are well understood. Theoretical analysis shows that stable FD methods with smooth coefficients converge at the rate predicted by truncation error as time and space steps go to zero. However, this theory does not apply to models of interest in seismic applications, since material properties that describe the subsurface are essentially discontinuous as functions of spatial location. Brown (1984); Gustafsson and Wahlund (2004) have shown that FD simulations in heterogeneous media are affected by two kinds of error. In addition to the error that corresponds to the truncation error of the homogeneous problem, there is another error that stems from inability of a FD scheme to resolve the position of the interface within a grid cell. The interface misalignment error is insensitive to the order of the scheme and can easily dominate overall error once grid dispersion is controlled. In practice this means that RMS error can be as large as 100% even when conventional grid-points-per-wavelength requirements are well satisfied (Symes and Vdovina, 2008; Symes et al., 2008). Reducing RMS error to 5% can require dramatic spatial oversampling, which suggests that FD schemes are unlikely to be cost effective for higher accuracy large-scale simulations.

Several methods that attempt to reduce interface error have been proposed over the years (Moczo et al., 2006; Zhang and LeVeque, 1997; Cohen and Joly, 1996). In this work, we consider finite element (FE) approach with mass lumping. By construction, coefficients of discrete systems resulting from FE approximations involve integration of the material properties over grid blocks and, thus, incorporate subgrid information about the medium. In the case of the constant density acoustics, this incorporation of the subgrid information is sufficient to eliminate first-order error due to sound velocity discontinuities. Symes (2009) proves that for constant density acoustics and smooth in time right-hand side, mass lumped FE method preserves optimal order of convergence in energy norm even if sound velocity is merely bounded and measurable. The justification is based on the fact that smoothness of solutions in time in case of constant density acoustics implies enough spatial regularity to give both optimal order of convergence of the FE approximation and the same order of convergence for the mass-lumped approximation. For the problems of interest in reflection seismology, such as simulation of seismograms, smoothness of solutions in time is a natural feature.

Our goal in this work is to demonstrate that FE method with mass lumping eliminates numerical artifacts due to the first order error at the cost equivalent to the cost of a FD method of the same order. We show that the FE scheme with mass lumping and numerical quadrature approximation of the stiffness matrix produces an explicit difference scheme, which is computationally equivalent to a regular grid FD approximation with specific spatial averaging of coefficients. We use two sound velocity models with interfaces not aligned with the numerical grid to demonstrate that lumped FE solutions have no evidence of stairstep diffractions in contrast to FD solutions. In the following sections, we introduce our model problem, discuss FD and FE methods employed in our study, and present numerical examples.

MODEL PROBLEM

We study the following 2D acoustic wave propagation problem in a constant density medium:

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 p}{\partial t^2}(\mathbf{x},t) - \nabla^2 p(\mathbf{x},t) = 0. \qquad (1)$$

The wave propagation is forced by initial conditions:

$$p(\mathbf{x},0) = p_0(\mathbf{x},0), \quad \frac{\partial p}{\partial t}(\mathbf{x},0) = \frac{\partial p_0}{\partial t}(\mathbf{x},0), \qquad (2)$$

where $p_0(\mathbf{x},t)$ is a solution of a homogeneous radiation problem for a point dilatational source. located at $\mathbf{x}_s$ given by Kirchhoff formula (Courant and Hilbert, 1962):

$$p_0(\mathbf{x},t) = \frac{1}{\pi}\int_0^{\sqrt{t-|\mathbf{x}-\mathbf{x}_s|/c}} \frac{g(t-|\mathbf{x}-\mathbf{x}_s|/c-\tau^2)}{\sqrt{\tau^2+2|\mathbf{x}-\mathbf{x}_s|/c}}\,d\tau, \qquad (3)$$

where $g(t)$ represents source time dependence function (“wavelet”).

FINITE DIFFERENCE METHOD

We have used FD discretizations of order two in time and $2K$, $K = 1,2,4$ in space to approximate equation (1). The following formula defines an approximation of the second derivative of a function f with respect to a

$$\frac{\partial^2 f}{\partial a^2}(a) \approx \frac{1}{\Delta a^2} \sum_{k=-K}^{K} \alpha_{|k|,K} f(a+k\Delta a), \qquad (4)$$

where coefficients $\alpha_{|k|,K}$ are chosen so that the order of the approximation is $O(\Delta a^{2K})$ (Cohen, 2002).

Regular grid FD schemes are obtained by replacing the time and space partial derivatives of $p(\mathbf{x},t)$ by approximations defined by equation (4), then requiring equation (1) to hold at $t = n\Delta t$, $x_i = m_i \Delta x_i$ for appropriate ranges of integers n and m_i, $i = 1,2$. Thus pressure p is defined at integer multiples of the spatial and time steps.

Regardless of the formal order of accuracy, FD methods applied to the wave problems with discontinuous material parameters are only first-order convergent (Brown, 1984; Gustafsson and Wahlund, 2004; Symes and Vdovina, 2008). The numerical error consists of two components. The higher-order component is responsible for the frequency-dependent grid dispersion and can be controlled by higher-order schemes. The other, first-order component is due to inaccurate interaction of the numerical wave with material interfaces and is insensitive to scheme order. Stairstep diffractions are widely known manifestations of the latter category of error.

FINITE ELEMENT METHOD

FE method is based on a weak formulation obtained by multiplying equation (1) by an arbitrary test function $v(\mathbf{x})$ and integrating over a wave propagation domain Ω (Ciarlet, 2002):

$$\int_\Omega \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} v(\mathbf{x}) + \int_\Omega \nabla p \cdot \nabla v(\mathbf{x}) = 0. \qquad (5)$$

We seek an approximate solution in a finite-dimensional space V_h in the form

$$p(\mathbf{x},t) \approx \sum_{j=1}^{N} \hat{p}_j(t) v_j(\mathbf{x}), \qquad (6)$$

where functions $v_j(\mathbf{x})$ constitute basis of V_h. We then substitute representation (6) into equation (5) and arrive at the following ODE system:

$$\mathbf{M}\frac{d^2\hat{\mathbf{p}}}{dt^2}(t) + \mathbf{S}\hat{\mathbf{p}}(t) = 0 \qquad (7)$$

with mass and stiffness matrices $\mathbf{M} = \{m_{ij}\}$ and $\mathbf{S} = \{s_{ij}\}$, $i,j = 1,\ldots,N$, and unknown vector $\hat{\mathbf{p}}(t) = [\hat{p}_1(t),\ldots,\hat{p}_N(t)]^{\mathrm{T}}$. The elements of the mass and stiffness matrices are computed by

$$m_{ij} = \int_\Omega \frac{1}{c^2(\mathbf{x})} v_i(\mathbf{x}) v_j(\mathbf{x}), \quad s_{ij} = \int_\Omega \nabla v_i(\mathbf{x}) \cdot \nabla v_j(\mathbf{x}). \qquad (8)$$

Similarly to FD method, the material discontinuities generally reduce the formal convergence rate of a FE method to the first order. However, in a special case of constant density acoustics, the optimal convergence rate is preserved.

MASS LUMPING

Solution of a non-trivial linear system arising from temporal discretization of equation (7) at every time step is prohibitively expensive for large-scale problems and can be avoided by mass lumping. Mass lumping consists of replacing mass matrix $\mathbf{M}$ by a diagonal matrix $\hat{\mathbf{M}} = \mathrm{diag}\{\hat{m}_1,\ldots,\hat{m}_N\}$. Standard lumping approach through application of Gauss-Lobatto quadrature rules with nodes coinciding with those of the nodal basis of FE method retains the order of accuracy in case of smooth coefficients, yet, fails to do so in case of the discontinuous medium (Cohen, 2002).

In an alternative mass lumping approach, the lumped matrix is obtained by summing the rows of the matrix $\mathbf{M}$:

$$\hat{m}_i = \sum_{j=1}^{N} m_{ij} = \int_\Omega \frac{1}{c^2(\mathbf{x})} v_i(\mathbf{x}). \qquad (9)$$

Symes (2009) considers bilinear (Q^1) elements and proves that in case of the constant density acoustics lumping via equation (9) preserves optimal order of convergence even when the sound velocity $c(\mathbf{x})$ is merely bounded and measurable.

IMPLEMENTATION

In this work, we have used Q^1 FE discretization with mass lumping via equation (9) and Gauss-Lobatto quadrature approximation of the stiffness matrix. This approach results in a difference scheme which is equivalent to a second-order in space and time FD scheme with spatially averaged coefficients (9) and produces pressure field free of stairstep diffractions.

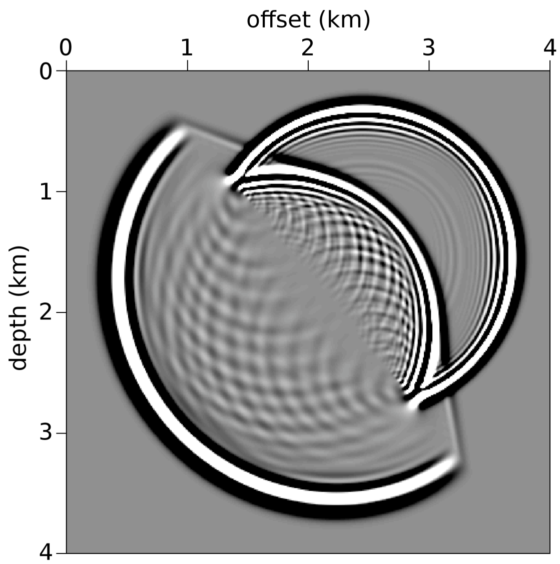
As with standard FD schemes, numerical dispersion of the method can by controlled by higher-order finite elements. However, application of higher-order FE methods results in a high-frequency error in the solution known as parasitic waves (Gustafsson et al., 1995). To avoid parasitic waves, we replaced quadrature approximation of the Q^1 stiffness matrix by higher-order FD discretizations of Laplacian. Numerical experiments confirmed second-order convergence of the resulting methods, and the expected suppression of grid dispersion with higher-order Laplacians.

NUMERICAL EXPERIMENTS

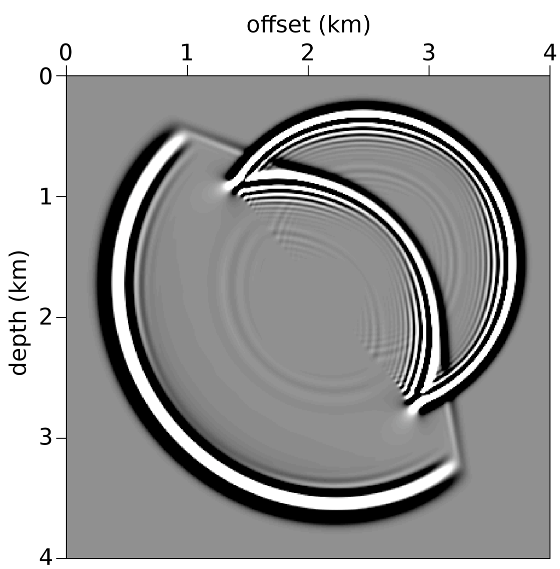
In this section we present two numerical examples based on a simple dipping interface model and a dome structure in 2D. Both models exhibit interfaces with considerable contrast, not aligned with the computational grid. We choose function $g(t)$ in equation (3) to produce a propagating Ricker wavelet with a peak frequency of 15 Hz.

In the first experiment, we assume a square domain of the size 4 km by 4 km discretized into 400×400 blocks and consider wave propagation for 0.7 s. We consider a dipping interface model with sound velocities of 1.5 km/s and 3 km/s in the upper and lower parts of the domain, respectively.

Figures 1(a) and 1(b) compare the results of a second-order in space and time FD and mass-lumped FE simulations on a 10 m grid. Figures 2(a) and 2(b) display the results of a fourth-order in space discretizations. As expected, FE solutions in Figures 1(b) and 2(b) show no evidence of the stairstep diffraction effect clearly visible in the FD solutions. From Figures 1(a) and 1(b) we see that both numerical solutions exhibit some grid dispersion, especially in the direct wave in the water. Numerical solutions shown in Figure 2 are much less dispersive due to the higher (fourth spatial) approximation order.



(a) FD solution, square 10 m grid.



(b) Mass lumped FE solution, square 10 m grid.

Figure 1: Comparison of pressure snapshots at 1.5 s of the second-order in time and space FD (top) and FE (bottom) solutions for Dipping Interface Model.

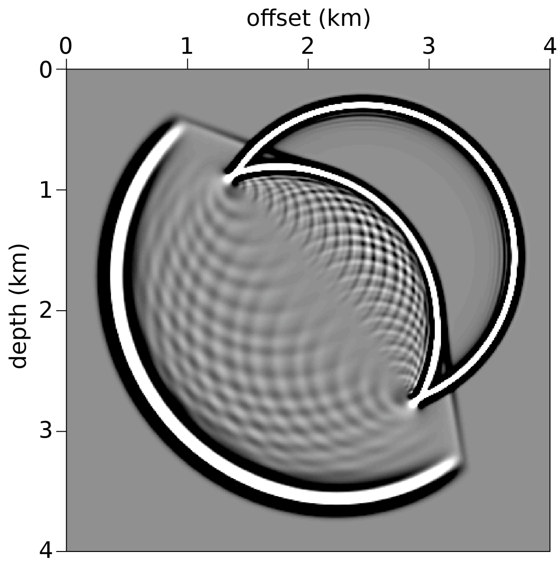
The sound velocity field used in the second experiment is displayed in Figure 3. Figures 4(a) and 4(b) present results of eighth-order in space FD and FE methods on a 10 m grid. As in the previous example, FD solution suffers from stairstep diffraction effect, which is particularly evident near the base of the dome between 4 and 6 km in depth. In contrast, FE solution shows no signs of stairstep diffraction.

CONCLUSION

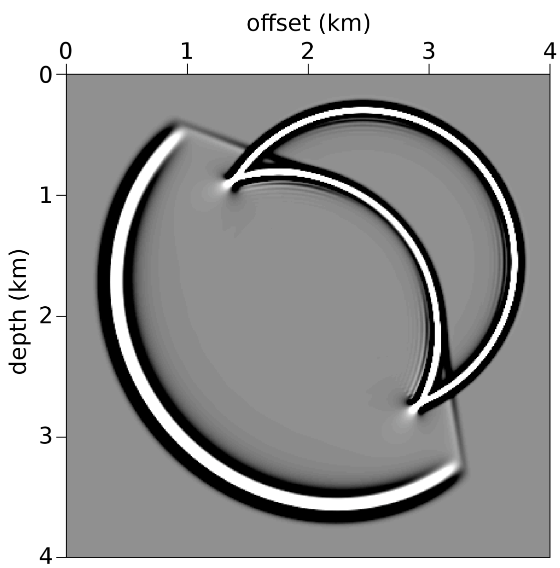
It is well known that in addition to higher-order dispersion error, FD simulation of waves in highly heterogeneous media is subject to the first-order interface misalignment error, which is insensitive to the order of the scheme. In this work, we show that in a special case of constant density acoustics, the Q^1 FE method with mass lumping retains the optimal second-order convergence rate even when sound velocity discontinuities are not aligned with the grid. Mass lumping combined with Gauss-Lobatto approximation of the stiffness matrix results in an explicit difference scheme, which is computationally equivalent to a second-order in time and $2K$ order in space FD scheme with stencil coefficients computed by a specific averaging rule. As with FD approximation, application of higher-order discretization of Laplacian controls the effect of numerical dispersion and prevents appearance of the parasitic waves. Our examples are consistent with formal mathematical analysis, and show that in case of the constant density acoustics the grid-interface misalignment error can be successfully eliminated by finite element approach. Stairstep diffraction, which is a well known manifestation of the first-order error in case of standard finite difference-based simulations is not present in our finite element-based numerical examples.

ACKNOWLEDGMENTS

The authors acknowledge with gratitude the support of the sponsors of The Rice Inversion Project. We are grateful to Tetyana Vdovina for helpful comments.



(a) FD solution, square 10 m grid.



(b) Mass lumped FE solution, square 10 m grid.

Figure 2: Comparison of pressure snapshots at 1.5 s of the second-order in time and fourth-order in space FD (top) and FE (bottom) solutions for Dipping Interface Model.

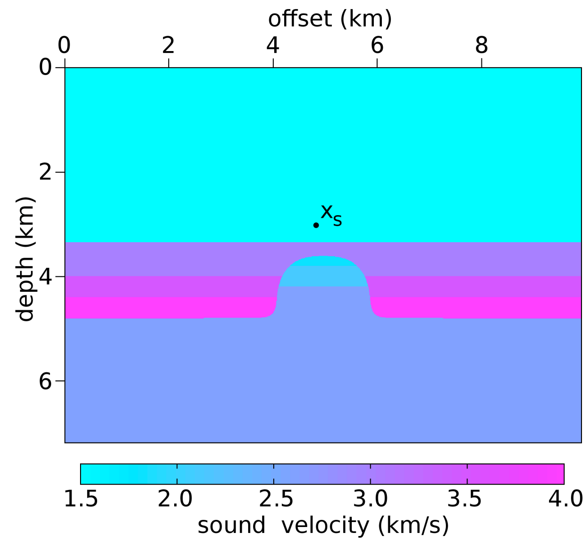
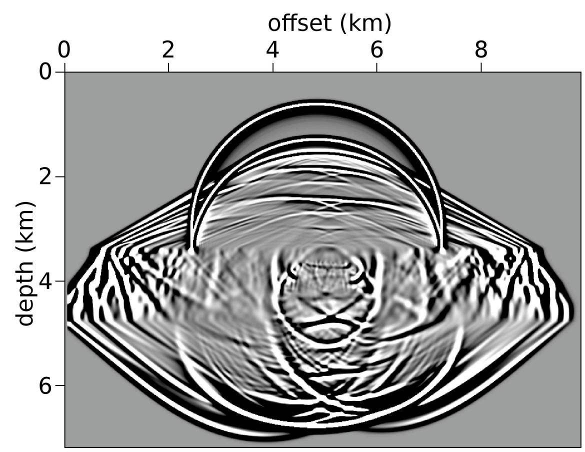
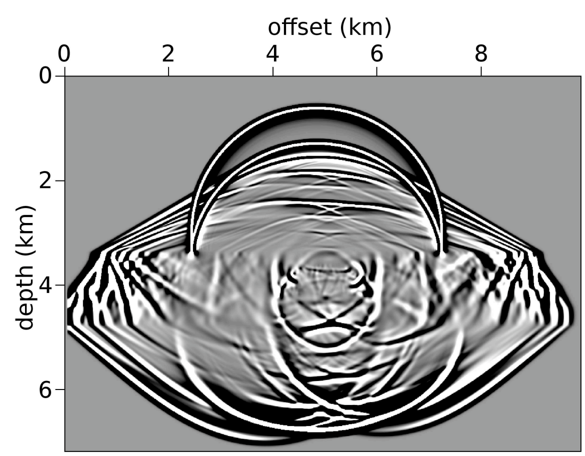


Figure 3: Sound velocity field for Dome Model. Source location is $\mathbf{x}_s = (3.011\,\text{km},\ 4.831\,\text{km})$.



(a) FD solution, square 10 m grid.



(b) Mass lumped FE solution, square 10 m grid.

Figure 4: Comparison of pressure snapshots at 0.7 s of the second-order in time and eighth-order in space FD (top) and FE (bottom) solutions for Dome Model.

EDITED REFERENCES

Note: This reference list is a copy-edited version of the reference list submitted by the author. Reference lists for the 2009 SEG Technical Program Expanded Abstracts have been copy edited so that references provided with the online metadata for each paper will achieve a high degree of linking to cited sources that appear on the Web.

REFERENCES

Brown, D., 1984, A note on the numerical solution of the wave equation with piecewise smooth coefficients: Mathematics of Computation, **42**, 369–391.

Ciarlet, P., 2002, The finite element method for elliptic problems: SIAM.

Cohen, G., and P. Joly, 1996, Construction and analysis of fourth oder finite difference schemes for the acoustic wave equation in nonhomogeneous media: Journal on Numerical Analysis, **33**, 1266–1302.

Cohen, G. C., 2002, Higher order numerical methods for transient wave equations: Springer.

Courant, R., and D. Hilbert, 1962, Methods of mathematical physics, vol. 2: Wiley-Interscience.

Gustafsson, B., H. O. Kreiss, and J. Oliger, 1995, Time dependent problems and difference methods: Wiley-Interscience.

Gustafsson, B. and P. Wahlund, 2004, Time compact high order difference methods for wave propagation, 2D: Journal of Scientific Computing, **25**, 195–211.

Moczo, P., J. O. A. Robertsson, and L. Eisner, 2006, The finite-difference time-domain method for modeling of seismic wave propagation: Advances in Geophysics, **48**, 421–516.

Symes, W., 2009, Mass lumping for constant density acoustics: Technical Report 09-10: Department of Computational and Applied Mathematics, Rice University.

Symes, W., I. Terentyev, and T. Vdovina, 2008, Gridding requirements for accurate finite difference simulation: 78th Annual International Meeting, SEG, 2077–2081.

Symes, W., and T. Vdovina, 2008, Interface error analysis for numerical wave propagation: Technical Report 08-22, Department of Computational and Applied Mathematics, Rice University.

Zhang, C., and R. J. LeVeque, 1997, The immersed interface method for acoustic wave equations with discontinuous coefficients: Wave Motion, **25**, 237–263.