Effective Media with a Continuum of Scales and Accurate FEM: Viewpoint of Owhadi and Zhang

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Background

Reflection-transmission at interface:

state-of-art is interface-adaptive unstructured mesh FEM - spectral element (Tromp-Komatitch) or DG (Käser-Dumbser, Hesthaven-Warburton - see Xin's talk later this AM).

Several averaging schemes proposed to rescue regular grid methods, based on effective medium theory (Muir et al. 1992) - hence *scale separation*.

However, typical distributions of elastic parameters in the earth show

- discontinuities, large and small, along interfaces of limited smoothness and spread throughout volumes
- apparent continuum of scales

Major new development: Owhadi's scale-free effective medium theory (Owhadi-Zhang 2006, 2008).

Accuracy and Approximation

Review of constant density case (illustration - Igor's talk):

$$\frac{1}{\kappa}\frac{\partial^2 p}{\partial t^2} - \nabla^2 p = f;\ \ p \equiv 0, t << 0$$

f is "low frequency" (well sampled in time - always the case)

$\Rightarrow$ pressure time derivatives have "finite energy", $1/\kappa$ bounded $\Rightarrow$ pressure Laplacian $\nabla^2 p$ has "finite energy" $\Rightarrow$ pressure has two L^2 derivatives

$\Rightarrow$ optimal approximation by Q^1 elements, hence $O(\Delta t)$ convergence of pressure derivatives, $O(\Delta t^2)$ convergence of pressure itself.

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Accuracy and Approximation

General case, discontinuous ρ:

$$\frac{1}{\kappa}\frac{\partial^2 p}{\partial t^2} - \nabla \cdot \frac{1}{\rho}\nabla p = f; \;\; p \equiv 0, t << 0$$

Displacement, hence acceleration $= \rho^{-1}\nabla p$ continuous

$\Rightarrow$ p has discontinuous derivatives where ρ is discontinuous $\Rightarrow$ no optimal order approximation by Q^1 elements $\Rightarrow$ no optimal order convergence (numerical evidence: Tommy's talk)

Crux of problem: how to create elements with appropriate approximation properties for p?

Smoothing via Change of Coordinates

Owhadi's observation:

Suppose that F is an invertible stationary coordinate transformation, $p(\mathbf{x}) = \tilde{p}(F(\mathbf{x}))$, that is, $p = \tilde{p} \circ F$. Then

$$\frac{\partial p}{\partial x_i} = \sum_j \frac{\partial F_j}{\partial x_i} \frac{\partial \tilde{p}}{\partial x_j} \circ F$$

so

$$\nabla \cdot \frac{1}{\rho} \nabla p = \sum_j [\nabla \cdot \frac{1}{\rho} \nabla F_j] \frac{\partial \tilde{p}}{\partial x_j} \circ F + \sum_{j,k} \left[\frac{1}{\rho} \nabla F_j \cdot \nabla F_k \right] \frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k} \circ F$$

Smoothing via Change of Coordinates

Set

$$a_{jk} = \left[\frac{1}{\rho}\nabla F_j \cdot \nabla F_k\right] \circ F^{-1}$$

and $\tilde{\kappa} = \kappa \circ F^{-1}$. Then $\tilde{p}$ solves

$$\frac{1}{\tilde{\kappa}}\frac{\partial^2 \tilde{p}}{\partial t^2} - \sum_{j,k} a_{jk}\frac{\partial^2 \tilde{p}}{\partial x_j \partial x_k} = f \circ F^{-1}$$

provided that the change of coordinates F is ρ-harmonic:

$$\nabla \cdot \frac{1}{\rho}\nabla F_j = 0.$$

On boundary of domain, make F the identity: $F_j = x_j$.

Smoothing via Change of Coordinates

What's really involved:

- the ρ-harmonic map F must be a change of coordinates: that is, continuous with a continuous inverse map
- the coefficient matrix a_{jk} must be *elliptic*, that is,

  $$a_{\min} I \leq a \leq a_{\max} I$$

  in the sense of symmetric matrices, for scalars $a_{\max} \geq a_{\min} > 0$ - actually more is necessary - "Cordes type conditions", see references.

Owhadi-Zhang 2006, 2008: generically OK in 2D, may fail for very high contrast in 3D.

Smoothing via Change of Coordinates

F is coordinate change, a_{jk} satisfies Cordes-type conditions

$\Rightarrow$ $\tilde{p}$ has two L^2 derivatives

$\Rightarrow$ Q^1 elements $\{\tilde{\phi}_j\}$ optimally approximate $\tilde{p}$

$\Rightarrow$ distorted Q^1 elements $\{\phi_j = \tilde{\phi}_j \circ F\}$ optimally approximate p

$\Rightarrow$ error in distorted Q^1 FE solution of wave equation $= O(\Delta t^2)$ - can also lump mass matrix using distorted elements.

Details - Tommy (next talk).

Perspective

Practical numerical method requires *localization* - construction of *global* ρ-harmonic coordinates too expensive.

Construction of distorted elements trivial in 1D for interface problems - what about 2D/3D? Probably needs to be *localized*. How accurately must F be computed?

Our observation: low (typical) density contrast $\Rightarrow$ little difference between ordinary, distorted Q^1 FEM.

What about elasticity? Or even acoustics in 1st order ("mixed FEM") formulation?

Perspective

Critical issue: how do we represent coefficients $\kappa, \rho, ...$?

Main lesson: simple grid sampling not enough - must somehow encode *subgrid* information (cf. also Tanya's upscaling work)

Proposal: multiscale representation (wavelets, curvelets, xxxxlets,...) via *oracle* - able to provide any average required with any precision required.

Meaning for inversion: produces estimates of averages, hence constraint on multiscale subgrid structure.

References: Houman Owhadi's web page, also Owhadi, H. and Zhang, L.: *Homogenization of the acoustic wave equation with a continuum of scales*, to appear in *Computer Methods in Applied Mechanics and Engineering*, 2008.

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