Well-posedness of Initial/Boundary Value Problems for Hyperbolic Integro-differential Systems with Nonsmooth Coefficients

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ABSTRACT

In the late 1960's, J.-L. Lions and collaborators showed that energy estimates could be used to establish existence, uniqueness, and continuous dependence on initial data for finite energy solutions of initial/boundary value problems for various linear partial differential evolution equations with nonsmooth coefficients. The second author has recently treated second order hyperbolic systems, for example linear elastodynamics, by similar methods, and extended these techniques to demonstrate continuous dependence and even differentiability (in a suitable sense) of the solution as function of the the coefficients. In the present paper, we extend Lions' results in a different direction, to first order symmetric hyperbolic integrodifferential systems (such as linear viscoelasticity) with bounded and measurable coefficients. We show that the initial value problem is well-posed in an appropriate space of finite-energy weak solutions. Solutions constructed by our method are continuous as functions of the coefficients and data. That this result is sharp: for example, solution are not in general locally uniformly continuous in coefficients and data. Solutions are however (Gâteaux-)differentiable as a function of the coefficients, in case the data is smooth enough that the time derivative of the solution is itself a finite-energy weak solution. The continuity result combines with the well-known domain of influence properties for hyperbolic systems with smooth coefficients to show that viscoelasticity with bounded, measureable coefficients predicts finite wave propagation speed.

INTRODUCTION

Continuum mechanics models propagation of small amplitude waves in fluids and solids by hyperbolic systems of linear partial differential or integrodifferential equations (Whitham, 1974; Gurtin, 1981). The coefficients appearing in these systems represent local (in space) continuum-mechanical characteristics of the material supporting the wave motion, such as the mass density or the elastic (Hooke) tensor components. While these coefficients might, in some cases, reasonably be modeled as smooth functions of position, in other cases they must be regarded as varying rapidly with position. For example, reasonably direct measurements (well logs) of density and elastic moduli in sedimentary rocks show substantial spatial variance at all measurable scales (Walden and Hosken, 1986; Bourbie et al., 1987; White et al., 1990). A fundamental hypothesis of continuum mechanics holds that bulk quantities depending on averages over finite volumes of material should be well-defined for any "reasonable" volume (sample masses, elastic moduli of samples as measured in the laboratory). Also, these bulk materical characteristics typically range over well-characterized intervals. Therefore a reasonable abstraction capturing these basic continuum–mechanical assumptions, while allowing for the observed spatial variability, might be that material properties (in the linear response regime) should be modeled as bounded and (Lebesgue-)measureable functions of position.

Beginning in the 1960's, J.-L. Lions and his collaborators provided a mathematical foundation for the study of time-dependent partial differential equations with bounded and measureable coefficients (Lions, 1971; Lions and Magenes, 1972). Lions established that several large classes of partial differential evolution equations have unique solutions depending continuously on their initial data, by exhibiting these problems as instances of a class of abstract evolution equations for which well-posedness could be demonstrated. These results imply, for example, that the linear acoustic wave equation has global (in time) finite energy solutions for bounded, measurable, and uniformly positive density and bulk modulus, under a variety of boundary conditions.

The second-named author of this paper recently extended Lion's approach to hyperbolic systems of second-order partial differential equations, including the linear elasticity system (Stolk, 2000). This work also showed that the solutions are smoothly dependent on the coefficients, in appropriate senses. These results provide a foundation for the study of inverse problems, in which the coefficients are to be determined or estimated from attributes of the solutions. A data-fitting approach to inverse problems via Newton's method and its relatives requires that derivatives of these attributes with respect to the coefficients can be defined (and computed). Thus smoothness of the solution as function of coefficients is a natural and fundamental question in the study of such problems.

In this paper, we extend the theory developed in (Lions, 1971; Lions and Magenes, 1972; Stolk, 2000) to a class of abstract first-order integro-differential systems. The coefficients of these systems are operators on an abstract Hilbert space. The first-order formulation encompasses several natural descriptions of wave propagation in continuum mechanics. The generalization to integro-differential systems permits us to treat models of wave propagation in materials with memory, such as variants of linear viscoelasticity. We show that systems of this class conform to a suitable generalization of the functional-analytic framework developed in (Lions, 1971; Lions and Magenes, 1972; Stolk, 2000). We define a class of weak solutions, and establish an *a priori* energy inequality for these. It follows from this *a priori* bound that *causal* weak solutions (vanishing for large negative time) are uniquely determined by the problem data (operator coefficients, right-hand side) and continuous in time. An abstract finite element method gives a direct construction of causal weak solutions. The restriction to causal solutions (distributions vanishing on a negative half-axis) is natural, in view of the integro-differential dynamics posited for our class of evolutions.

Examples from continuum mechanics (acoustics, elastodynamics) suggest appropriate Hilbert spaces of mechanical states (spatial distributions of stresses, particle velocities, etc.) with the norm-squared being essentially mechanical energy. We refer throughout the paper to an appropriate weighted norm-squared as "energy", for this reason.

This theory also suggests a natural sense in which causal weak solutions are continuous as functions of the coefficient operators and data. We show that convergence of weak solutions of the abstract evolution problem, locally uniformly in time, follows from strong (pointwise) convergence of the coefficient operators. Moreover, weak solutions are differentiable as a functions of the coefficients, provided that the data is differentiable in time (so that time derivative of solutions are themselves finite-energy solutions of similar problems). The energy estimates already entail continuity as function of data, so these results actually imply that weak solutions are continuous as functions of coefficients and data jointly, and even differentiable provided that the data is sufficiently regular. Additional regularity of the data is certainly required for differentiability: we observe that, without it, weak solutions are not, in general, even locally uniformly continuous as functions of coefficients and data.

In view of the memory effects modeled by our abstract system, causal solutions form the natural class uniquely determined by the problem data. If memory effects are absent, so that the problem is differential, then initial (plus other problem) data uniquely determine solutions. We sketch the modifications of the theory necessary to accommodate such initial value problems.

We conclude by showing explicitly how certain models of viscoelastic wave propagation fit into this framework. The conditions imposed on coefficient operators translate into (minimal) regularity requirements on the material parameters of the model. Mass density and viscoelastic moduli are required to be bounded and measurable as functions on the spatial domain of the problem, and elliptic in a suitable sense. We observe that strong convergence of the coefficient operators is a consequence of *convergence in measure* of the material parameter functions. One immediate consequence of this fact is the finite propagation speed of viscoelastic waves in materials models with merely bounded and measurable density and moduli: this conclusion follows directly from the continuity result, the approximation in measure of bounded measurable functions by smooth functions, and the finite propagation speed of similar systems with smooth coefficients.

DEFINITION OF THE PROBLEM

Let H be a separable real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Denote by $\mathcal{B}(H)$ the Banach space of continuous linear operators on H, with the operator (uniform) norm. We suppose that

- $A \in \mathcal{B}(H)$ is self-adjoint and positive-definite;
- $B \in \mathcal{B}(H)$;
- $Q \in L^1(\mathbf{R}, \mathcal{B}(H))$, and is *causal*, that is, $Q(t) = 0$ for $t < 0$;
- D is a skew-adjoint operator with dense domain $V \subset H$.

When convenient, we metrize V with the graph norm of D.

Define the bounded operator $R : L^2(\mathbf{R}, H) \to L^2(\mathbf{R}, H)$ by

$$R[u](t) = \int Q(t-s)u(s)\, ds. \tag{1}$$

Note that if $\operatorname{supp} u \subset [T, \infty)$ for some $T \in \mathbf{R}$, then $\operatorname{supp} R[u] \subset [T, \infty)$ also. The formal (distribution) adjoint

$$R^*[u](t) = \int Q(s-t)u(s)\, ds$$

satisfies a similar condition: if $\operatorname{supp} u \subset (-\infty, T]$ for some $T \in \mathbf{R}$, then $\operatorname{supp} R^*[u] \subset (-\infty, T]$.

The components described above combine to yield the *formal evolution problem*: find u which solves, in a suitable sense,

$$Au' + Du + Bu + R[u] = f \in L^2(\mathbf{R}, H) \tag{2}$$

Example. Linear acoustics provides an important example of the framework just described. Acoustic wave propagation does not include the memory effect modeled by the integral term (operator R), but illustrates several other features of the class of problems studied in this paper.

We presume that the fluid supporting acoustic wave motion occupies a domain $\Omega \subset \mathbf{R}^3$ with rectifiable boundary. The balance and constitutive laws of linear acoustics relate the excess pressure $p(t, \mathbf{x})$ and velocity fluctuations $\mathbf{v}(t, \mathbf{x}) = (v_1(\mathbf{x}, t), v_2(\mathbf{x}, t), v_3(\mathbf{x}, t))^T$, $\mathbf{x} \in \mathbf{R}^3$, to mass density $\rho(\mathbf{x})$, bulk modulus $\kappa(\mathbf{x})$, and body force density $\mathbf{f}(t, \mathbf{x})$ by

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} &= -\nabla p + \mathbf{f}, \\ \frac{1}{\kappa} \frac{\partial p}{\partial t} &= -\nabla \cdot \mathbf{v}. \end{aligned} \tag{3}$$

Define $H = (L^2(\Omega))^4$. Then

$$Au = \operatorname{diag}\left(\frac{1}{\kappa}, \rho, \rho, \rho\right) u, \ u = \begin{pmatrix} p \\ v_1 \\ v_2 \\ v_3 \end{pmatrix} \in H$$

defines a bounded self-adjoint positive-definite operator $A \in \mathcal{B}(H)$, provided that $\log \rho, \log \kappa \in L^\infty(\Omega)$.

Define the differential operator

$$D = -\begin{pmatrix} 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 & 0 \end{pmatrix}.$$

Then D defines a skew-adjoint operator on H with dense domain

$$V = H_0^1(\Omega) \times H_{\text{div}}^1(\Omega).$$

The Hilbert space $H_{\text{div}}^1(\Omega)$ is the dense subspace of $(L^2(\Omega))^3$ obtained by completing $C^1(\Omega)$ in the graph norm of the divergence operator.

With these choices for the spaces H and V and operators A, D, $B = 0$, $Q \equiv 0$, and $f \in L^2(\mathbf{R}, H)$ defined by

$$f(t) = \mathbf{f}(t, \cdot),$$

the acoustics system (3) is formally equivalent to the evolution problem (2). If the material parameter distributions κ and/or ρ are not smooth (of class C^1 at least), however, then the form of the equations (3) immediately implies that no solutions of class C^1 may exist, even if the right-hand side f (that is, the body force density $\mathbf{f}$) is smooth. Physically reasonable fluid configurations thus exist for which solutions in the classical sense cannot be defined, for example piecewise homogeneous mixtures with jump discontinuities of density and/or bulk modulus across smooth interfaces. A more flexible notion of "solution" than the classical (C^1) type is required to treat such problems.

We follow (Lions, 1971; Lions and Magenes, 1972; Stolk, 2000) in defining *weak solutions* in $L^2_{\text{loc}}(\mathbf{R}, H)$ of the formal evolution problem (2) by integration against smooth test functions. Because the operator kernel Q may have unbounded support, we must constrain the growth of candidate members of $L^2_{\text{loc}}(\mathbf{R}, H)$ on the negative half-axis. Accordingly, a *weak solution* of the formal evolution problem (2) is a member of $u \in L^2_{\text{loc}}(\mathbf{R}, H)$ satisfying

1. For every $T \in \mathbf{R}$, $u \in L^2((-\infty, T], H)$;

2. $$\int \langle u(t), (A\phi' + D\phi - B^*\phi - R^*[\phi])(t) \rangle \, dt = -\int \langle f(t), \phi(t) \rangle \, dt; \tag{4}$$

   for all $\phi \in C_0^\infty(\mathbf{R}, V)$.

Note that since $R^*[\phi]$ is supported in the half-axis $(-\infty, \sup \operatorname{supp} \phi]$, and is square-integrable, assumption 1. implies that the last term on the left-hand side of (4) is well-defined.

A weak solution is *causal* if there exists $T_0 \in \mathbf{R}$ so that $u = 0$ in $(-\infty, T_0)$. Because of the causal assumption on the convolution kernel Q, existence of a causal weak solution vanishing for $t < T_0$ implies that the right-hand side f is also causal, in fact vanishes for $t < T_0$.

Note that a causal weak solution u belongs to $L^2((-\infty, T], H)$ for any $T \in \mathbf{R}$, whence $R[u] \in L^2_{\mathrm{loc}}(\mathbf{R}, H)$ is well-defined and $R[u] \in L^2((-\infty, T], H)$ for any $T \in \mathbf{R}$.

We shall repeatedly use following property of weak solutions:

Theorem 1. *Suppose that $u \in L^2_{\mathrm{loc}}(\mathbf{R}, H)$ is a weak solution of (2). Then for any $\eta \in C_0^\infty(\mathbf{R})$,*

$$\eta * u \in C^\infty(\mathbf{R}, V). \tag{5}$$

Remark: The content of this theorem is that smoothing in time also "smooths in space", in the sense that the values of the smoothed weak solution are confined to the subspace V.

Proof. Choose the test function ϕ in (4) to have the special form $\phi(s) = \eta(t-s)w$, where $t \in \mathbf{R}$, $w \in V$, and $\eta \in C_0^\infty(\mathbf{R})$. Then

$$\begin{aligned}
\langle \eta * u(t), Dw \rangle &= \left\langle \int \eta(t-s)u(s)\, ds, Dw \right\rangle \\
&= \int \langle u(s), D(\eta(t-s)w) \rangle\, ds \\
&= \int \Big[\langle \bar{u}(s), -A\eta'(t-s)w + B^*\eta(t-s)w \\
&\quad + R^*[\eta(t-\cdot)w](s) \rangle \Big]\, ds - \langle \eta * f(t), w \rangle
\end{aligned} \tag{6}$$

where the last equality is simply a rearrangement of (4) with the special choice of test function $\phi(s) = \eta(t-s)w$ mentioned above. The right-hand side of (6) is bounded by a w-independent multiple of $\|w\|_H$, therefore so is the left. Therefore $\eta * \bar{u}$ takes values in the domain $\mathcal{D}(D^*)$ of the adjoint D^* for any $\eta \in C_0^\infty(\mathbf{R})$. But D is skew-adjoint, so $\mathcal{D}(D^*) = \mathcal{D}(D) = V$. □

THE ENERGY INEQUALITY

Define the *energy* $E(t)$ of a weak solution u of (2) by

$$E(t) = \frac{1}{2}\langle u(t), Au(t) \rangle \tag{7}$$

It follows from the definition of weak solution that E is well-defined almost everywhere, and locally integrable. Because A is positive-definite,

$$C_* \|u(t)\|^2 \le E(t) \le C^* \|u(t)\|^2 \tag{8}$$

hold for almost all $t \in \mathbf{R}$, for suitable $C^* \ge C_* > 0$.

Remark. In the linear acoustics example presented in the previous section, $E(t)$ is precisely the mechanical energy of the acoustic field at time t.

In this and the following sections, we will use C to denote a generic nonnegative constant depending on $C_*, C^*, \|B\|_{\mathcal{B}(H)}$, and $\|Q\|_{L^1(\mathbf{R},\mathcal{B}(H))}$, and possibly on other quantities as noted.

Theorem 2. *Let $u \in L^2_{\rm loc}(\mathbf{R}, H)$ be a weak solution of (2), $E \in L^1_{\rm loc}(\mathbf{R})$ its energy as defined in (7). Then*

- *after modification on a set of measure zero, E is continuous;*
- *if in addition u is causal, then for any $T \in \mathbf{R}$ there exists $C \geq 0$ so that for $t \in (-\infty, T]$,*

$$E(t) \leq C \int_{-\infty}^{t} \|f\|^2. \tag{9}$$

Proof. Let $\eta_n \in C_c^\infty(\mathbf{R})$ be an approximate identity, that is, $\eta_n(t) = n\eta(nt)$, where

$$\eta \in C_0^\infty(\mathbf{R}),\ \eta \geq 0,\ \int \eta(t)\, dt = 1,\ \text{supp}\ \eta \subset [-1, 1]. \tag{10}$$

Define

$$E_n(t) = \frac{1}{2}\langle \eta_n * u(t), A(\eta_n * u)(t)\rangle. \tag{11}$$

Since $\eta_n * u \to u$ in $L^2_{\rm loc}(\mathbf{R}, H)$, $E_n \to E$ in $L^1_{\rm loc}(\mathbf{R})$.

For each n, E_n is smooth; differentiating E_n, obtain for any $s, t \in \mathbf{R}$

$$\begin{aligned}
E_n(t) - E_n(s) &= \int_s^t \frac{dE_n}{ds}(\tau)\, d\tau \\
&= \int_s^t \langle \eta_n' * u(\tau), A(\eta_n * u)(\tau)\rangle\, d\tau \\
&= \int_s^t \left\langle \int \eta_n'(\tau - \sigma) u(\sigma)\, d\sigma, A(\eta_n * u)(\tau) \right\rangle d\tau \\
&= \int_s^t \int \left\langle u(\sigma), -A\frac{d}{d\sigma}\left[\eta_n(\tau - \sigma)(\eta_n * u)(\tau)\right] \right\rangle d\sigma d\tau.
\end{aligned} \tag{12}$$

The inner (σ) integral has the form of the first term in (4), with test function $\sigma \mapsto \eta_n(\tau - \sigma)(\eta_n * u)(\tau)$. Thanks to Theorem 1, this function lies in $C_0^\infty(\mathbf{R}, V)$, whence (4) implies that the right-hand side of (12) is

$$
\begin{aligned}
&= \int_s^t \int \langle u(\sigma), (D - B^*)(\eta_n(\tau - \sigma)(\eta_n * u)(\tau)) \\
&\quad -R^*[\eta_n(\tau - \cdot)(\eta_n * u)(\tau)](\sigma)\rangle \, d\sigma d\tau \\
&\quad + \int_s^t \int \langle f(\sigma), \eta_n(\tau - \sigma)(\eta_n * u)(\tau)\rangle \, d\sigma d\tau \\
&= \int_s^t [-\langle((D + B)(\eta_n * u)(\tau) + R[\eta_n * u])(\tau), \eta_n * u(\tau)\rangle \\
&\quad + \langle \eta_n * f(\tau), \eta_n * u(\tau)\rangle] \, d\tau \\
&= \int_s^t [-\langle(B(\eta_n * u)(\tau) + R[\eta_n * u])(\tau), \eta_n * u(\tau)\rangle \\
&\quad + \langle \eta_n * f(\tau), \eta_n * u(\tau)\rangle] \, d\tau. \qquad (13)
\end{aligned}
$$

The last equality in this sequence is a consequence of the skew-symmetry of D.

Since convolution with η commutes with the convolution operator R, and with the actions of the other operators appearing in (2), the identity (13) implies that

$$
\begin{aligned}
|E_n(t) - E_n(s)| &= \left| \int_s^t \frac{dE_n}{d\tau}(\tau)\, d\tau \right| \\
&\leq \int_s^t \Big[|\langle \eta_n * (Bu(\cdot))(\tau), \eta_n * u(\tau)\rangle| \\
&\quad + |\langle(\eta_n * R[u])(\tau), \eta_n * u(\tau)\rangle| \\
&\quad + |\langle \eta_n * f(\tau), \eta_n * u(\tau)\rangle| \Big] \, d\tau \\
&\leq (\|B\| + 1) \left(\int_s^t \|(\eta_n * u)(\tau)\|^2 \, d\tau \right) \\
&\quad + \int_s^t (\|(\eta_n * R[u])(\tau)\|^2 + \|(\eta_n * f)(\tau)\|^2) \, d\tau \qquad (14)
\end{aligned}
$$

Since u, $R[u]$, and f are locally square-integrable, for each $t \in \mathbf{R}$ and $\epsilon > 0$, there exist $\Delta t(t, \epsilon) > 0$ and an $N(t, \epsilon) \in \mathbf{N}$ so that for $|s - t| < \Delta t(t, \epsilon)$ and $n > N(t, \epsilon)$,

$$
\int_{s-1/n}^{t+1/n} \|u\|^2 < \epsilon, \ \int_{s-1/n}^{t+1/n} \|R[u]\|^2 < \epsilon, \ \text{and} \ \int_{s-1/n}^{t+1/n} \|f\|^2 < \epsilon,
$$

whence (14) implies that for $n > N(t, \epsilon)$, $|s - t| < \Delta t(t, \epsilon)$,

$$
|E_n(t) - E_n(s)| < C\epsilon. \qquad (15)
$$

Continuity of E_n implies existence of $\overline{\Delta t}(t, \epsilon) > 0$ so that for $|s - t| < \overline{\Delta t}(t, \epsilon)$ and $n \leq N$, $|E_n(t) - E_n(s)| < C\epsilon$. Thus the inequality (15) holds for all $n \in \mathbf{N}$ if s satisfies $|s-t| < \min(\Delta t(t, \epsilon), \overline{\Delta t}(t, \epsilon))$. Since $t \in \mathbf{R}, \epsilon > 0$ are arbitrary, we have shown that the sequence $\{E_n\} \subset C^0(\mathbf{R})$ is equicontinuous.

Choose $T_0 \le T \in \mathbf{R}$: it follows from inequality (14) and Young's inequality that

$$\int_{T_0}^{T} E_n \le C \int_{T_0-1}^{T+1} \|u\|^2 \tag{16}$$

is bounded independently of n. For $t \in [T_0, T]$,

$$(T - T_0)E_n(t) = \int_{T_0}^{T} (E_n(t) - E_n(s))\, ds + \int_{T_0}^{T} E_n(s)\, ds \tag{17}$$

Apply Young's inequality to the convolutions with η_n appearing in the inequality (14) to conclude that for $T_0 \le s, t \le T$,

$$|E_n(t) - E_n(s)| \le C \int_{T_0-1}^{T+1} (\|u\|^2 + \|R[u]\|^2 + \|f\|^2). \tag{18}$$

Taken together, (16), (17) and (18) imply that $\{E_n\}$ is a bounded subset of $C^0([T_0, T])$. According to Ascoli's theorem, $\{E_n\}$ is precompact in $C^0([T_0, T])$, hence has a subsequence converging uniformly to a continuous limit. Since the subsequence is necessarily also L^1-convergent, and $T \in \mathbf{R}$ is arbitrary, the first assertion of the theorem is established.

In view of the continuity of E, we may take the limit $n \to \infty$ on both sides of the inequality (14) along the uniformly convergent subsequence whose existence we have just established. Since $\eta_n * u \to u$ in $L^2([T_0, T], H)$, the right hand side converges, and we obtain

$$|E(t) - E(s)| \le C \int_{s}^{t} (\|u\|^2 + \|R[u]\|^2 + \|f\|^2). \tag{19}$$

We have assumed u to be causal, but this assumption has not appeared in the reasoning up to now. It allows us to take $s \to -\infty$ in (19). In view of the equivalence of $\sqrt{E}$ and the norm $\|\cdot\|$ (inequalities (8)), the inequality (19) implies that

$$E(t) \le C \int_{-\infty}^{t} (E + \|f\|^2).$$

Gronwall's inequality then yields the second conclusion. $\square$

Corollary 1. *The energy E of a weak solution u of* (2)*, as defined above, satisfies for any $s, t \in \mathbf{R}$*

$$E(t) - E(s) = \int_{s}^{t} \langle -Bu(\tau) - R[u](\tau) + f(\tau), u(\tau)\rangle\, d\tau. \tag{20}$$

Proof. Continuity of E and convergence of $\eta_n * u$ to u in $L^2_{\mathrm{loc}}(\mathbf{R}, H)$ allows us to take limits on both sides of (13). $\square$

Corollary 2. *Suppose that $u_1, u_2 \in L^2_{\mathrm{loc}}(\mathbf{R}, H)$ are* causal *weak solutions of (2). Then $u_1 = u_2$.*

Proof. The conclusion follows immediately from the energy inequality (9), applied to the difference $u = u_1 - u_2$, which is a weak solution with $f \equiv 0$. $\square$

Corollary 3. . *Suppose that $u \in L^2_{\text{loc}}(\mathbf{R}, H)$ is a causal weak solution of (2). Then $u \in C^0(\mathbf{R}, H)$.*

Proof. For $\delta t \in \mathbf{R}$, denote by $u_{\delta t}$ the member of $L^2_{\text{loc}}(\mathbf{R}, H)$ defined by $u_{\delta t}(t) = u(t + \delta t)$. Then $u_{\delta t}$ is a causal weak solution (the only one, thanks to Corollary 2) of (2) with f replaced by $f_{\delta t} \in L^2(\mathbf{R}, H)$, defined by $f_{\delta t}(t) = f(t + \delta t)$. The translation group acts strongly continuously on L^2, i.e. $\|f_{\delta t} - f\|_{L^2(\mathbf{R},H)} \to 0$ as $\delta t \to 0$. Since the difference $u_{\delta t} - u$ is a causal solution of (2) with right-hand side $f_{\delta t} - f$, it follows immediately from (9) that $\|u_{\delta t}(t) - u(t)\|_H \to 0$ as $\delta t \to 0$ for any $t \in \mathbf{R}$, that is, $u \in C^0(\mathbf{R})$. $\square$

Corollary 4. *Suppose that*

1. *$\mathcal{K} \subset \mathcal{B}(H)$ is a bounded set;*
2. *$\mathcal{L} \subset \mathcal{B}(V, H)$ is a bounded set of skew-adjoint operators on H with (common) domain V, whose graph norms are all equivalent (to each other and to the norm in V);*
3. *$\mathcal{M} \subset \mathcal{B}(H)$ is a bounded set of self-adjoint, uniformly positive definite operators: there exist constants $0 < C_* \leq C^*$ so that for all $A \in \mathcal{M}$,*
$$C_* I \leq A \leq C^* I;$$
4. *$\mathcal{Q} \subset L^1(\mathbf{R}, \mathcal{B}(H)) \bigcap C^0(\mathbf{R}_+, \mathcal{B}(H))$ is a bounded set of causal operator-valued functions: if $Q \in \mathcal{Q}$, then $Q(t) = 0$ for $t < 0$.*

Let the set $\mathcal{P} \subset \mathcal{M} \times \mathcal{L} \times \mathcal{K} \times \mathcal{Q}$ parametrize a family of formal evolution problems of for (2), with coefficients $A \in \mathcal{M}, D \in \mathcal{L}, B \in \mathcal{K}$, and $Q \in \mathcal{Q}$, with common right-hand side $f \in L^2(\mathbf{R}, H)$, and let $\mathcal{U} \subset L^2_{\text{loc}}(\mathbf{R}, H)$ be a corresponding family of causal weak solutions. Then $\mathcal{U} \subset C^0(\mathbf{R}, H)$ is equicontinuous.

Proof. That $\mathcal{U} \subset C^0(\mathbf{R}, H)$ is the content of the last Corollary. It follows from the proof of the basic energy estimate (9) that the constant C appearing in its right-hand side may be chosen uniform over $\mathcal{P}$ - indeed, the bounds defining the sets listed in conditions 1-4 above are precisely those on which our constants, canonically notated C, depend. Therefore (9) implies that for $u \in \mathcal{U}$,

$$\|u(t + \delta t) - u(t)\|^2 \leq \frac{1}{C_*} E_{u_{\delta t} - u}(t) \leq C \int_{-\infty}^{t} \|f_{\delta t} - f\|^2 = C \int_{t}^{t+\delta t} \|f\|^2$$

from which a uniform modulus of continuity follows. $\square$

EXISTENCE OF WEAK SOLUTIONS

The proof of existence follows the pattern laid out by Lions (1971), which in turn echos Cauchy's proof of the fundamental theorem of ordinary differential equations. We define a Galerkin method, show that it converges, and finally that the limit is a weak solution. Note that no rate of convergence follows from this argument; in fact it is easy to see that none can be expected.

Theorem 3. *Assume that $f \in L^2(\mathbf{R}, H)$ is causal:* $\operatorname{supp} f \subset [T_0, \infty)$ *for some $T_0 \in \mathbf{R}$, and that the causal convolution kernel $Q \in L^1(\mathbf{R}, \mathcal{B}(H))$ is continuous in $\mathbf{R}_+$: $Q \in C^0(\mathbf{R}_+, \mathcal{B}(H))$. Then a unique causal weak solution u of (2) exists, and* $\operatorname{supp} u \subset [T_0, \infty)$.

Proof. In view of the energy estimate, which establishes $L^2(\mathbf{R}, H)$-continuous dependence of weak solutions on the right hand side f, it suffices to establish existence of solutions for a $L^2(\mathbf{R}, H)$-dense set of right hand sides. Therefore assume without loss of generality that in addition $f \in C^0(\mathbf{R}, H)$.

Since H is separable and $V \subset H$ is dense, countable linearly independent subsets $\{w_k\}_{k=1}^{\infty} \subset V$ exist for which finite linear combinations are dense in H. Without loss of generality, assume that $\{w_k\}_{k=1}^{\infty}$ is (H-) orthonormal: $\langle w_k, w_l \rangle = \delta_{kl}$, $k, l \in \mathbf{N}$.

Define $m \times m$ matrices A^m (symmetric positive definite), D^m and B^m by

$$A^m_{kl} = \langle A w_k, w_l \rangle, \tag{21}$$

$$D^m_{kl} = \langle D w_k, w_l \rangle, \tag{22}$$

$$B^m_{kl} = \langle B w_k, w_l \rangle, \tag{23}$$

for $1 \leq k, l \leq m$, and the operator R^m on $L^2_{\mathrm{loc}}(\mathbf{R})^m$ defined analogously to (1) by

$$R^m U^m(t) = \int_{-\infty}^{t} \langle Q^m(t-s) U^m(s)\, ds,\ Q^m_{kl}(t) \rangle\, ds = \langle Q(t) w_k, w_l \rangle,\ 1 \leq k, l \leq m.$$

Note that $Q^m \in L^1(\mathbf{R}, \mathcal{B}(\mathbf{R}^m)) \bigcap C^0(\mathbf{R}_+, \mathcal{B}(\mathbf{R}^m))$ is causal ($Q^m(t) = 0, t < 0$).

Define $F^m \in C^0(\mathbf{R})^m$ by

$$F^m_k(t) = \langle f(t), w_k \rangle,\ 1 \leq k \leq m.$$

A minor modification of a standard contraction mapping argument (see for example Coddington and Levinson (1955)) shows that for each $m \in \mathbf{N}$, the initial value problem

$$\begin{aligned} A^m \frac{dU^m}{dt} + D^m U^m + B^m U^m + R^m U^m &= F^m, \\ U^m(t) &= 0, \ t < T_0. \end{aligned} \tag{24}$$

has a unique solution $U^m \in C^1(\mathbf{R}, \mathbf{R}^m)$.

For each $m \in \mathbf{N}$, define $u_m \in C^1(\mathbf{R}, V)$, $f_m \in C^0(\mathbf{R}, H)$ by

$$u_m(t) = \sum_{k=1}^{m} U_k^m(t) w_k, \ \ f_m(t) = \sum_{k=1}^{m} F_k^m(t) w_k.$$

Then the system (24) satisfied by U^m, together with the H-orthonormality of $\{w_k\}$, implies that u_m is the weak solution of the evolution equation (2) with right-hand side f_m. The energy estimate (9) shows that the sequence u_m is bounded in $L^2_{\mathrm{loc}}(\mathbf{R}, H)$, hence by the Tychonoff-Alaoglu theorem and a diagonal process argument weakly precompact in $L^2_{\mathrm{loc}}(\mathbf{R}, H)$. Denote by $u_{m(l)}$ a weakly convergent subsequence, and by u its weak limit. Since $u_m(t) = 0$ for $t < T_0$ and all $m \in \mathbf{N}$, the same is true for u.

To see that the limit u is a weak solution of (2), introduce for each $m_0 \in \mathbf{N}$ test functions ψ of the form

$$\psi = \sum_{k=1}^{m_0} \phi_k \otimes w_k, \quad \phi_k \in C_0^\infty(\mathbf{R}). \tag{25}$$

For l sufficiently large that $m(l) > m_0$, $\langle f^m(t), \psi(t) \rangle = \langle f(t), \psi(t) \rangle$, $\langle D u_{m(l)}, \psi \rangle = -\langle u_{m(l)}, D\psi \rangle$, etc. So

$$\int \langle u_{m(l)}, A\psi' + D\psi - B^*\psi - R^*[\psi] \rangle \, dt = -\int \langle f, \psi \rangle \, dt.$$

Letting $l \to \infty$, it follows that u satisfies (4) for all test functions ψ of the form given in equation (25). Since linear combinations of w_m's are dense in H, the set of functions of the form (25) is dense in $C_0^1(\mathbf{R}, V)$, whence u is a weak solution of (2). $\square$

Additional regularity of the right-hand side f translates into additional regularity of the solution.

Corollary 5. *Suppose that $f \in H^k(\mathbf{R}, H)$, $k \geq 0$, is causal. Then the unique causal weak solution u of (2) satisfies $u \in H^k_{\mathrm{loc}}(\mathbf{R}, H)$.*

Proof. By induction on k: the case $k = 0$ is Theorem 3. Denote by v the weak solution of the problem (2) with right-hand side f'. By induction, $v \in H^{k-1}_{\text{loc}}(\mathbf{R}, H)$. From Theorem 3, v is causal, so

$$u(t) = \int_{-\infty}^{t} v$$

is well-defined. It is straightforward to see that u is a weak solution of (2) with right-hand side f, and that $u \in H^k_{\text{loc}}(\mathbf{R}, H)$ as claimed. $\square$

CONTINUOUS DEPENDENCE ON PARAMETERS

As we will now study a suite of problems of the form (2), it is convenient to choose a fixed Hilbert space structure for the dense subspace $V \subset H$. The skew-adjoint operators D figuring in the abstract evolution problem (2) are assumed to be bounded $V \to H$. That is, the norm in V is equivalent to the graph norm of D.

Theorem 4. *Suppose that*

1. $\mathcal{K} \subset \mathcal{B}(H)$ *is a bounded set;*
2. $\mathcal{L} \subset \mathcal{B}(V, H)$ *is a bounded set of skew-adjoint operators on H with (common) domain V, whose graph norms are all equivalent (to each other and to the norm in V);*
3. $\mathcal{M} \subset \mathcal{B}(H)$ *is a bounded set of self-adjoint, uniformly positive definite operators: there exist constants* $0 < C_* \leq C^*$ *so that for all* $A \in \mathcal{M}$,

$$C_* I \leq A \leq C^* I;$$

4. $\mathcal{Q} \subset L^1(\mathbf{R}, \mathcal{B}(H)) \bigcap C^0(\mathbf{R}_+, \mathcal{B}(H))$ *is a bounded set of causal operator-valued functions: if* $Q \in \mathcal{Q}$, *then* $Q(t) = 0$ *for* $t < 0$.

Assume that the sequence of problems of the form (2) with coefficients A_m, D_m, B_m, Q_m approximates the problem with coefficients A, D, B, Q in the sense that

1. $\{A_m : m \in \mathbf{N}\} \subset \mathcal{M}$, $A \in \mathcal{M}$, $\lim_{m\to\infty} \|(A_m - A)w\| \to 0$ *for all* $w \in H$;
2. $\{D_m : m \in \mathbf{N}\} \subset \mathcal{L}$, $D \in \mathcal{L}$, $\lim_{m\to\infty} \|(D_m - D)v\| \to 0$ *for all* $v \in V$;

3. $\{B_m : m \in \mathbf{N}\} \subset \mathcal{K}$, $B \in \mathcal{K}$, $\lim_{m\to\infty} \|(B_m - B)w\| \to 0$, *for all* $w \in H$;

4. $\{Q_m : m \in \mathbf{N}\} \subset \mathcal{Q}$, $Q \in \mathcal{Q}$, *and the convolution operators* R_m, R *with kernels* Q_m, Q *satisfy* $\lim_{m\to\infty} \|R_m[w] - R[w]\|_{L^2(\mathbf{R},H)} \to 0$ *for all* $w \in L^2(\mathbf{R}, H)$.

Let u_m*, respectively* u*, be causal weak solutions of the differential equation* (2) *with coefficients* (A_m, D_m, B_m, Q_m)*, respectively* (A, D, B, Q)*. and (uniform) right-hand side* $f \in L^2(\mathbf{R}, H)$*. Then* $u_m \to u$ *strongly in* $L^2_{\text{loc}}(\mathbf{R}, H)$*, that is,* $\|u_m - u\|_{L^2((-\infty,T],H)} \to 0$ *for all* $T \in \mathbf{R}$.

This theorem will be proven by showing that the assumptions imply first that u_m converges to u weakly in $L^2_{\text{loc}}(\mathbf{R}, H)$. This convergence result then implies that u_m converges to u weakly, pointwise in t. These two results, along with the original assumption on convergence of the operators, finally gives strong L^2 convergence.

Lemma 1. *Under the conditions of Theroem 4,* u_m *converges weakly to* u *in* $L^2_{\text{loc}}(\mathbf{R}, H)$.

Remark. In order that R_m converge to R pointwise, as assumed in the statement of the preceding theorem, it is sufficient that $Q_m \to Q$ uniformly in $\mathbf{R}_+$.

Proof. The bounds implied by the energy estimate (Theorem 2) are uniform over bounded sets of coefficients as described in the statement of the theorem. Therefore $\{u_m\}$ is bounded in $L^2_{\text{loc}}(\mathbf{R}, H)$, hence has a $L^2_{\text{loc}}(\mathbf{R}, H)$-weakly convergent subsequence $u_{m(l)}$, with limit $\bar{u} \in L^2_{\text{loc}}(\mathbf{R}, H)$. Note that $L^2_{\text{loc}}(\mathbf{R}, H)$-weak convergence implies convergence in the sense of H-valued distributions on $\mathbf{R}$. Choose a test function $\phi \in C_0^\infty(\mathbf{R}, V)$: then

$$
\begin{aligned}
-\int \langle f(s), \phi(s)\rangle \, ds &= \int \langle u_{m(l)}(s), (A_{m(l)}\phi' + D_{m(l)}\phi - B^*_{m(l)}\phi - R^*_{m(l)}[\phi])(s)\rangle \, ds && (26)\\
&= \int \langle \bar{u}(s), (A\phi' + D\phi - B^*\phi - R^*[\phi])(s)\rangle \, ds && (27)\\
&\quad + \int \langle (u_{m(l)}(s) - \bar{u}(s)), (A\phi' + D\phi - B^*\phi - R^*[\phi])(s)\rangle \, ds && (28)\\
&\quad + \int \Big\langle u_{m(l)}(s), ((A_{m(l)} - A)\phi' + (D_{m(l)} - D)\phi && (29)\\
&\quad -(B_{m(l)} - B)^*\phi - (R_{m(l)} - R)^*[\phi])(s) \Big\rangle \, ds && (30)
\end{aligned}
$$

The second term vanishes in the limit $l \to \infty$ because of the weak convergence of $u_{m(l)}$ to $\bar{u}$. The coefficients $A_m, ...$ range over bounded sets of operators, so we may replace ϕ and ϕ' in the third term with simple V-valued functions, taking finitely many values, at the price of an arbitrarily small perturbation in this term, uniformly in l. However the strong convergence of the coefficient operators assumed in the statement of the theorem then implies that the resulting integrals become arbitrarily small as $l \to \infty$. Thus $\bar{u}$ is a weak solution of the problem (2), and must therefore be the same as the (unique) weak solution u constructed in the preceding section. Thus no other weak accumulation point of the bounded sequence $\{u_m\}$ may exist, hence $u_m \rightharpoonup u$ in $L^2_{\text{loc}}(\mathbf{R}, H)$ as claimed. $\square$

Corollary 6. *Under the conditions of Theorem 4, u_m converges to u weakly, pointwise in t: that is, $u_m(t) \rightharpoonup u(t)$ for all $t \in \mathbf{R}$.*

Proof. According to Corollary 4, the conditions described in the statement of Theorem 4 imply that $\{u_m : m \in \mathbf{N}\}$ is equicontinuous. Given $t \in \mathbf{R}$ and $\epsilon > 0$, choose $\Delta t > 0$ so that if $|\delta t| < \Delta t$,

$$\|u_m(t+\delta t) - u_m(t)\| < \epsilon, \ m \in \mathbf{N}; \ \|u(t+\delta t) - u(t)\| < \epsilon.$$

Then

$$\left\| u_m(t) - \frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} u_m \right\| < \epsilon, \ m \in \mathbf{N}; \left\| u(t) - \frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} u \right\| < \epsilon.$$

However, according to Lemma 1, for any $w \in H$,

$$\frac{1}{2\Delta t}\int_{t-\Delta t}^{t+\Delta t} \langle u_m - u, w\rangle = \int \left\langle u_m - u, w\frac{1}{2\Delta t}\mathbf{1}_{[t-\Delta t, t+\Delta t]}\right\rangle \to 0, \ m \to \infty.$$

Therefore, assuming without loss of generality that $\|w\| = 1$,

$$|\langle u_m(t) - u(t), w\rangle| \le 3\epsilon$$

for m sufficiently large. Since $\epsilon > 0$ is arbitrary, the proof is complete. $\square$

Proof of Theorem 4. A brief calculation gives

$$\begin{aligned}\langle u_m, A_m u_m\rangle - \langle u, Au\rangle &= \langle u_m - u, A_m(u_m - u)\rangle \\ &\quad + \langle 2u_m - u, (A_m - A)u\rangle + 2\langle u_m - u, Au\rangle. \end{aligned} \tag{31}$$

The right-hand side f in the formal evolution equation (2) for both u_m and u vanishes for sufficiently large negative t, else u could not be causal, but then u_m and u must vanish on a common (negative) half-axis, thanks to Corollary 2. The energy identity (20) implies that

$$\begin{aligned}
&\langle u_m, A_m u_m\rangle - \langle u, Au\rangle(t) \\
&= -\int_{-\infty}^{t} \langle B_m u_m, u_m\rangle - \langle Bu, u\rangle + \langle R_m[u_m], u_m\rangle - \langle R[u], u\rangle - \langle f, u_m - u\rangle \\
&= -\int_{-\infty}^{t} \langle u_m - u, B_m(u_m - u)\rangle + \langle 2u_m - u, (B_m - B)u\rangle + 2\langle u_m - u, Bu\rangle \\
&\quad -\int_{-\infty}^{t} \langle u_m - u, R_m[u_m - u]\rangle + \langle 2u_m - u, R_m[u] - R[u]\rangle + 2\langle u_m - u, R[u]\rangle \\
&\quad +\int_{-\infty}^{t} \langle f, u_m - u\rangle. \qquad (32)
\end{aligned}$$

Identities (31) and (32) combine to yield

$$\langle u_m - u, A_m(u_m - u)\rangle(t) = -\int_{-\infty}^{t} \langle B_m(u_m - u) + R_m[u_m - u], u_m - u\rangle + g_m, \qquad (33)$$

in which $g_m \in C^0(\mathbf{R})$ is defined by

$$\begin{aligned}
g_m(t) \;=\; & -\langle 2u_m - u, (A_m - A)u\rangle - 2\langle u_m - u, Au\rangle \\
& -\int_{-\infty}^{t} \langle 2u_m - u, (B_m - B)u\rangle + 2\langle u_m - u, Bu\rangle \\
& -\int_{-\infty}^{t} \langle 2u_m - u, R_m[u] - R[u]\rangle + 2\langle u_m - u, R[u]\rangle + \int_{-\infty}^{t} \langle f, u_m - u\rangle. \qquad (34)
\end{aligned}$$

Since the B_m's are uniformly bounded operators on H and the R_m's are uniformly bounded operators on $L^2((-\infty, t], H)$ for every $t \in \mathbf{R}$ (with norm independent of t), (33) implies that

$$\begin{aligned}
&\langle u_m - u, A_m(u_m - u)\rangle(t) \\
&\leq C\int_{-\infty}^{t} (\|u_m - u\|(\tau) + \|u_m - u\|_{L^2((-\infty,\tau],H)}(\tau))\, d\tau + |g_m(t)|. \qquad (35)
\end{aligned}$$

Select T_0 for which $u(t) = u_m(t) = 0$ for all $t < T_0, m \in \mathbf{N}$. Then the uniform positivity of the A_m's ((8), also Lemma 1, assumption 3) combines with (35) to yield

$$\langle u_m - u, A_m(u_m - u)\rangle(t) \leq C(1 + (t - T_0)) \int_{T_0}^{t} \langle u_m - u, A_m(u_m - u)\rangle(\tau)\, d\tau + |g_m(t)|. \qquad (36)$$

Choose $T \in \mathbf{R}$. Application of Gronwall's inequality to (36) yields, for $T_0 \leq t \leq T$ and C depending on T along with everything else,

$$\langle u_m - u, A_m(u_m - u)\rangle \leq e^{C(1+(T-T_0))} \int_{T_0}^{T} |g_m|. \qquad (37)$$

As mentioned in the proof of Lemma 1, $\{u_m(t) : m \in \mathbf{N}\}$ is bounded for each t, whence $\{u_m : m \in \mathbf{N}\}$ is bounded in $L^2([T_0, T], H)$. This observation implies immediately that $\{g_m(t) : t \in$

$[T_0, T], m \in \mathbf{N}\}$ is a bounded subset of $\mathbf{R}$.The assumptions on the convergence $A_m \to A$, $B_m \to B$ and $R_m \to R$, imply that the terms of $g_m(t)$ involving $(A_m - A)$, $(B_m - B)$ and $R_m[u] - R[u]$ vanish pointwise in the limit $m \to \infty$. The other terms tend to zero pointwise by the weak convergence of $u_m - u$, pointwise in t and in $L^2((-\infty, T], H)$. Thus $\{g_m\}$ is uniformly bounded on $[T_0, T]$, and convergent pointwise to zero. The Dominated Convergence Theorem shows that the integral on the right hand side of (37) vanishes in the limit $m \to \infty$, therefore so does the left-hand side. Integrating (37) from T_0 to T we see that also $\int_{T_0}^{T} \langle u_m - u, A_m(u_m - u)\rangle \to 0$. In view of (8), valid uniformly in m, the proof is complete. $\square$

Remark. This result is sharp, in the sense that nothing stronger than continuity can be expected without additional constraints on the various components of the formal evolution problem (2). In particular, the modulus of continuity cannot be uniform in the right-hand side ($f \in L^2(\mathbf{R}, H)$), even locally.

For example, the 1D linear advection problem

$$\left(\frac{1}{c}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\right)(t, x) = f(t, x)$$

conforms to the setting described above, with $H = L^2(\mathbf{R})$. The operator coefficients are: $A =$ multiplication by the positive constant $1/c$, $D = \partial/\partial x$, skew-adjoint with domain $V = H^1(\mathbf{R})$, and $B \equiv 0, Q \equiv 0$. For any $f \in L^2(\mathbf{R}^2)$ ($\equiv L^2(\mathbf{R}, H)$ by Fubini's Theorem), the causal weak solution is

$$u[c, f](t, x) = c\int_{-\infty}^{t} f(\tau, x + c(t - \tau))d\tau, \tag{38}$$

in which we have explicitly indicated the dependence of the weak solution on the coefficient $1/c$ and the right-hand side f.

Suppose $\chi \in C_0^\infty(\mathbf{R})$, $\operatorname{supp}\chi \subset [-1, 1]$, and

$$\int \chi = 1.$$

For $\epsilon > 0$, set $f_\epsilon(t, x) = \cos((x + t)/\epsilon)\chi(x + t)\chi(x)$. Then $u[1, f_\epsilon](t, x) = \cos((x + t)/\epsilon)\chi(x + t)$ for $t > 1$, whereas a straightforward application of the method of stationary phase shows that

$$u[c, f_\epsilon](t, x) = O\left(\sqrt{\frac{\epsilon}{|c - 1|}}\right)$$

for $c \neq 1$. Thus the modulus of continuity of $c \mapsto u[c, f_\epsilon](t, \cdot) \in H$ (for $t > 1$) cannot be uniform over the bounded set $\{f_\epsilon : \epsilon > 0\} \subset L^2(\mathbf{R}, H)$, as c ranges over any interval containing 1.

On the other hand, additional regularity of the right-hand side entails more regular behaviour of the solution, as one might expect.

Corollary 7. *In addition to the hypotheses of Theorem 4, assume that* $f \in H^k(\mathbf{R}, H)$*,* $k \geq 0$*. Then* $\|u_m - u\|_{H^k((-\infty,T],H)} \to 0$ *as* $m \to \infty$ *for any* $T \in \mathbf{R}$*.*

Proof. Follows directly from Theorem 4 and its proof, and from Corollary 5. □

Remark. For systems such as linear acoustics and linear viscoelastodynamics, described elsewhere in this paper, in which the coefficient operators act by multiplication with matrix-valued functions, sufficient conditions for operator approximation may be stated in terms of the coefficient functions. L^∞-bounded L^1 convergence of coefficients induces strong convergence of the corresponding operators. For example, smoothing the density and bulk modulus of the acoustic system induces approximation of the pressure and velocity fields. In the final section we will explicitly state a continuity theorem for the dependence of viscoelastic stress and velocity fields on the density and viscoelastic moduli, for example.

Considerably weaker approximation of the coefficients may also lead to convergence in energy, if the weaker sense of coefficient approximation is compensated by additional regularity of the right-hand side. The first such results, so far as we know, were established by Bamberger et al. (1977, 1979) for the 1D 2nd order scalar wave equation. This "dynamic homogenization" approach has been extended very recently to the 2D and 3D scalar wave equation, under some restrictions; see Ohwadi and Zhang (2006).

The examples presented elsewhere in this paper define families of evolution problems sharing a common skew-adjoint operator D. The next theorem shows that for such problems, the solution is actually differentiable as a function of the remaining coefficients, provided that the right-hand side possesses a minimal amount of additional regularity.

Theorem 5. *Suppose that the operators* A, D, B*, and* $\{Q(t) : t \in \mathbf{R}\}$ *satisfy the conditions of Theorem 4, and* $f \in H^1(\mathbf{R}, H)$ *is causal. Denote by* $u \in H^1_{\text{loc}}(\mathbf{R}, H)$ *the causal weak solution of* (2) *with these choices of coefficients and right-hand side, per Theorem 3 and Corollary 5. Assume that* $\delta A, \delta B \in \mathcal{B}(H)$*,* δA *is self-adjoint, and* $\delta Q \in L^1(\mathbf{R}, \mathcal{B}(H)) \bigcap C^0(\mathbf{R}_+, \mathcal{B}(H))$*. Define for* $h \in \mathbf{R}$

$$A_h = A + h\delta A,\ B_h = B + h\delta B,\ Q_h = Q + h\delta Q.$$

For sufficiently small h, A_h is self-adjoint and positive definite, so that the problem (2) *with coefficients A_h, D, B_h, Q_h and right-hand side f has a unique weak solution $u_h \in H^1_{\mathrm{loc}}(\mathbf{R}, H)$ (Corollary 5). Denote by $\delta u \in L^2_{\mathrm{loc}}(\mathbf{R}, H)$ the weak solution of the formal evolution problem*

$$A\delta u' + D\delta u + B\delta u + R[\delta u] = -\delta A u' - \delta B u - \delta R[u], \tag{39}$$

in which R (δR) is the convolution operator with kernel Q (δQ), as usual. Then

$$\left\| \frac{u_h - u}{h} - \delta u \right\|_{L^2((-\infty,T],H)} = o_h(1). \tag{40}$$

for any $T \in \mathbf{R}$.

Proof. The meaning of (39) is that for any $\phi \in C_0^\infty(\mathbf{R}, V)$,

$$\begin{aligned} \int \langle \delta u, A\phi' + D\phi - B^*\phi - R^*[\phi] \rangle &= \int \langle \delta A u' + \delta B u + \delta R[u], \phi \rangle \\ &= -\int \langle u, \delta A \phi' - \delta B^* \phi - \delta R^*[\phi] \rangle. \end{aligned} \tag{41}$$

On the other hand, both u and u_h, $h > 0$, satisfy (4) with the same right-hand side, so

$$\begin{aligned} 0 &= \frac{1}{h}\left(\int \langle u_h, A_h\phi' + D\phi - B_h^*\phi - R_h^*[\phi] \rangle \right. \\ &\quad \left. - \int \langle u, A\phi' + D\phi - B^*\phi - R^*[\phi] \rangle \right) \\ &= \int \langle u_h, \delta A\phi' - \delta B^*\phi - \delta R^*[\phi] \rangle \\ &\quad + \int \left\langle \frac{u_h - u}{h}, A\phi' + D\phi - B^*\phi - R^*[\phi] \right\rangle. \end{aligned} \tag{42}$$

Subtracting (41) from (42) and rearranging, obtain

$$\begin{aligned} \int \left\langle \left(\frac{u_h - u}{h} - \delta u \right), A\phi' + D\phi - B^*\phi - R^*[\phi] \right\rangle &= \int \langle u_h - u, \delta A\phi' - \delta B^*\phi - \delta R^*[\phi] \rangle \\ &= -\int \langle \delta A(u_h - u)' + \delta B(u_h - u) + \delta R[u_h - u], \phi \rangle. \end{aligned} \tag{43}$$

In view of equation (43), the Newton quotient remainder

$$\frac{u_h - u}{h} - \delta u$$

is the weak solution of (2) with right-hand side

$$\delta A(u_h - u)' + \delta B(u_h - u) + \delta R[u_h - u] \in L^2_{\mathrm{loc}}(\mathbf{R}, H).$$

In view of Corollary 7,

$$\|\delta A(u_h - u)' + \delta B(u_h - u) + \delta R[u_h - u]\|_{L^2((-\infty,T],H)} \to 0$$

as $h \to 0$ for any $T \in \mathbf{R}$. Now a simple cutoff argument and Theorem 2 yield (40). $\square$

Remark. This result is also sharp, in the sense that the right-hand side must have at least one square-integrable derivative in t, if only additional regularity in t is to be imposed. For example, the solution (38) of the linear advection equation presented above may be rewritten as

$$u(t,x) = \int_x^\infty f\left(t + \frac{x-y}{c}, y\right)\, dy,$$

from which it is straightforward to see that no less regularity in t will do. On the other hand, the expression (38) suggests that additional regularity in x might also support differentiable dependence on c. However this conclusion rests on a special feature of the example problem, namely that it admits a propagation of singularity principle (and indeed solution via the method of characteristics, an even more special property). Propagation of singularities along bicharacteristics holds for symmetric or strictly hyperbolic systems with smooth coefficients (see for example Taylor (1981)), and to some limited extent for systems with less regular coefficients (Beals and Reed, 1982, 1984; Symes, 1986; Lewis and Symes, 1991; Bao and Symes, 1996). Stronger regularity results for dependence on coefficients follow for some of these systems. The matter seems worthy of further study.

THE DIFFERENTIAL CASE

If the memory term (convolution operator R) is absent, then initial data determine solutions uniquely. In this section, we sketch the theory, parallel to that for causal solutions, which holds in this differential case. We assume throughout this section that $Q \equiv 0$.

Corollary 8. *Suppose that $u \in L^2_{\mathrm{loc}}(\mathbf{R}, H)$ is a weak solution of (2). Then $u \in C^0(\mathbf{R}, H)$.*

Proof. Choose $\phi \in C^\infty(\mathbf{R})$ so that $\phi(t) = 1$ for $t > 1$ (say), and $\phi(t) = 0$ for $t < -1$. Set $u_+ = \phi u$, $u_- = (1-\phi)u$. It is straightforward to verify that u_+ is a causal solution of (2) with f replaced by $f + \phi' u \in L^2(\mathbf{R}, H)$, whence $u_+ \in C^0(\mathbf{R}, H)$ according to Corollary3. Likewise $t \mapsto u_-(-t)$ is also a causal solution of (2) with D, B replaced by $-D$, $-B$, and f replaced by $t \mapsto -f(-t) + \phi' u(-t)$, which also belongs to $L^2(\mathbf{R}, H)$. Thus u_- is also continuous, but $u = u_+ + u_-$. $\square$

Corollary 9. *Suppose that u is a weak solution of* (2)*. Then for any $s \le t \in \mathbf{R}$,*

$$E(t) \le E(s) + C \int_s^t \|f\|^2.$$

Proof. The energy identity (20) applies to weak solutions, causal or not. Take into account $R = 0$, and use Gronwall's inequality, the boundedness of B, and the equivalence (8) of the energy with the norm in H. □

Corollary 10. *If u_1 and u_2 are weak solutions of* (2) *(with the same right-hand side $f \in L^2(\mathbf{R}, H)$), and $u_1(s) = u_2(s)$ for some $s \in \mathbf{R}$, then $u_1 \equiv u_2$.*

Proof. Set $u = u_1 - u_2$: u is a weak solution with right-hand side $f = 0$, and $u(s) = 0$. The result follows immediately from Corollary 9. □

Theorem 6. *Suppose that $T_0 \in \mathbf{R}$ and $u_0 \in H$. Then there exists a unique weak solution of* (2) *for which $u(T_0) = u_0$.*

Proof. A proof of this result is precisely analogous to the proof of Theorem 3: the solution is approximated by a Galerkin procedure and the solution of systems of ordinary differential equations, and the energy in the error estimated (in this instance, via Corollary 9). □

Results precisely analogous to those established in the last section hold concerning regular dependence on the coefficient operators for weak solutions with specified initial data and right-hand side. We leave the reader to formulate these results, whose proofs are minor variants of those given above.

AN EXAMPLE: VISCOELASTICICTY

The dynamic equations of linear viscoelasticity may be written as

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} &= \nabla \cdot \sigma + \mathbf{f}, \\ \Gamma * \frac{\partial \sigma}{\partial t} &= \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T). \end{aligned} \tag{44}$$

in which $\mathbf{v}$ is the particle velocity field, σ the stress tensor, $\mathbf{f}$ a body force density, ρ the mass density, and Γ the inverse Hooke operator (Christensen, 1983; Pipkin, 1986). Viscoelasticity differs from elasticity in that the inverse Hooke operator is a convolution operator (in time), rather than a temporally local multiplication operator. It is necessarily causal, to enforce causality in the system response. It follows from (44) that the strain rate (right-hand side of the second equation) is the convolution of the stress with the indefinite time integral of Γ. Instantaneous elastic response, which we shall assume, requires that a nonzero strain rate arise immediately from a stress impulse; therefore the kernel Γ can be decomposed as

$$\Gamma(t) = \Gamma^e \delta(t) + \gamma(t),$$

in which Γ^e is the elastic inverse Hooke tensor (inverse of the unrelaxed modulus), and γ is a causal kernel. Both the elastic kernel Γ^e and the memory kernel $\gamma(t)$ act on symmetric tensor fields by spatially-variable, symmetry-preserving linear operators. The conventional representation of such things by 4-index tensors,

$$\Gamma^e = \left(\Gamma^e_{ijkl}\right)\Big|^3_{i,j,k,l=1},\ \gamma = \left(\gamma_{ijkl}\right)\Big|^3_{i,j,k,l=1},$$

thus entail the symmetries

$$\Gamma^e_{ijkl} = \Gamma^e_{jikl} = \Gamma^e_{ijlk} = \Gamma^e_{klij},\ i,j,k,l = 1,2,3, \tag{45}$$

and similarly for γ.

These fields are permitted to vary in space. To avoid technical complications, assume that the viscoelastic material occupies all of $\mathbf{R}^3$. We require that, for some $0 < g_* \leq g^*$,

1. Γ^e is elliptic: for any symmetric $\sigma \in \mathbf{R}^{3\times 3}$,

$$g_* \|\sigma\| \leq \|\Gamma^e(\mathbf{x})\sigma\| \leq g^* \|\sigma\|,\ \mathbf{x} \in \mathbf{R}^3; \tag{46}$$

2. $\Gamma^e \in L^\infty(\mathbf{R}^3, \mathcal{B}(\mathbf{R}^{3\times 3}_{\text{symm}}))$;

3. $\gamma \in W^{1,1}(\mathbf{R}, L^\infty(\mathbf{R}^3, \mathcal{B}(\mathbf{R}^{3\times 3}_{\text{symm}})))$.

For the "state space" H of the viscoelastic system we choose $H = L^2(\mathbf{R}^3, \mathbf{R}^9) \equiv L^2(\mathbf{R}^3, \mathbf{R}^{3\times 3}_{\text{symm}}) \times L^2(\mathbf{R}^3, \mathbf{R}^3)$. The inner product in H is defined by

$$\langle u_1, u_2 \rangle = \int_{\mathbf{R}^3} \operatorname{tr} \sigma_1^T \sigma_2 + \mathbf{v}_1^T \mathbf{v}_2,\ u = \begin{pmatrix} \sigma \\ \mathbf{v} \end{pmatrix}.$$

The assumptions 1-3 above and the symmetries (45) imply that

$$Au = \begin{pmatrix} \Gamma^e \sigma \\ \\ \rho \mathbf{v} \end{pmatrix}, \; u = \begin{pmatrix} \sigma \\ \\ \mathbf{v} \end{pmatrix} \in H$$

defines a bounded, self-adjoint positive-definite operator $A \in \mathcal{B}(H)$.

Define the differential operator $D : C_0^\infty(\mathbf{R}^3, \mathbf{R}^9) \to C_0^\infty(\mathbf{R}^3, \mathbf{R}^9)$ by

$$Du = -\begin{pmatrix} \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T) \\ \\ \nabla \cdot \sigma \end{pmatrix}, \; u = \begin{pmatrix} \sigma \\ \\ \mathbf{v} \end{pmatrix} \in C_0^\infty(\mathbf{R}^3, \mathbf{R}^9) \equiv C_0^\infty(\mathbf{R}^3, \mathbf{R}^{3\times 3}_{\mathrm{symm}}) \times C_0^\infty(\mathbf{R}^3, \mathbf{R}^3).$$

D is antisymmetric and densely defined in H. Denote its skew-adjoint extension also by D, and the domain of the extension by V.

Let

$$b = \lim_{t \to 0^+} \gamma(t, \cdot) \in L^\infty(\mathbf{R}^3, \mathcal{B}(\mathbf{R}^{3\times 3}_{\mathrm{symm}})),$$

and

$$q = \lim_{t \to 0^+} \mathbf{1}_{[t,\infty)} \frac{\partial \gamma}{\partial t} \in L^1(\mathbf{R}, L^\infty(\mathbf{R}^3, \mathcal{B}(\mathbf{R}^{3\times 3}_{\mathrm{symm}}))).$$

Then

$$\gamma * \frac{\partial \sigma}{\partial t} = b\sigma + q * \sigma.$$

Define $B \in \mathcal{B}(H)$ and $Q \in L^1(\mathbf{R}, \mathcal{B}(H))$ by

$$Bu = \begin{pmatrix} b\sigma \\ \\ 0 \end{pmatrix}, \; Q(t)u = \begin{pmatrix} q(t)\sigma \\ \\ 0 \end{pmatrix}, \; u = \begin{pmatrix} \sigma \\ \\ \mathbf{v} \end{pmatrix}$$

Finally, define $f \in L^2(\mathbf{R}, H)$ by $f = (0, \mathbf{f})^T$, which implicitly presumes that $\mathbf{f} \in L^2(\mathbf{R}^4, \mathbf{R}^3) \equiv L^2(\mathbf{R}, L^2(\mathbf{R}^3, \mathbf{R}^3)$.

With these definitions, the system (44) is formally equivalent to the evolution problem (2). The theory developed here thus assures the existence of weak solutions of (44), in material models including discontinuities of densities and/or elastic moduli and/or relaxation moduli.

To see that these solutions exhibit finite propagation speed, hence might reasonably be called waves, we consider first the case of smooth coefficients. The following theorem and corollary express minor variants of a well-known results about hyperbolic systems with smooth coefficients (see for example Lax (2006), Ch. 4), and we shall omit the proofs:

Theorem 7. *In the formulation of the formal evolution problem* (2)*, suppose that* $H = L^2(\mathbf{R}^n)^p$*, and that in addition to the hypotheses outlined in the first section, the operators* A, B *represent multiplication by smooth* $p \times p$ *matrix-valued functions, and use the same letters to denote these functions. Assume that* Q *is a smooth* $p \times p$ *matrix-valued function on* $\mathbf{R}^{n+1}$*, absolutely integrable as a function of its first coordinate with values in* $L^\infty(\mathbf{R}^n)^{p\times p} \bigcap C^\infty(\mathbf{R}^n)^{p\times p}$*. Assume also that* D *takes the form*

$$Du = \sum_{i=1}^{n} K_i \frac{\partial u}{\partial x_i},\ K_i \in \mathbf{R}^{p\times p},\ K_i^T = K_i,\ i = 1, ..., n, \tag{47}$$

and that $f \in C_0^\infty(\mathbf{R}^{n+1})^p$*. Then*

1. *the causal weak solution* $u \in L^2_{\text{loc}}(\mathbf{R}, H)$ *is smooth:* $u \in C^\infty(\mathbf{R}^{n+1})^p$*, and*

2. *if* $\phi \in C^\infty(\mathbf{R}^n)$ *satisfies*
$$A + \sum K_i \frac{\partial \phi}{\partial x_i} > 0$$
and $\text{supp}(f) \bigcap \{(\mathbf{x}, t) : \phi(\mathbf{x}) > t\} = \emptyset$*, then* $u(\mathbf{x}, t) = 0$ *if* $\phi(\mathbf{x}) \le t$*.*

Corollary 11. *In the setting of Theorem 7, suppose that* $\tau \in \mathbf{R}$ *satisfies*

$$\tau A(\mathbf{x}) + \sum_{i=1}^{n} K_i \xi_i \ge 0,\ \mathbf{x} \in \mathbf{R}^n,\ |\xi| = 1. \tag{48}$$

If $\mathbf{x}_0 \in \mathbf{R}^n, t_0 \in \mathbf{R}$ *satisfy*

$$f(\mathbf{x}, t) = 0 \text{ if } \tau|\mathbf{x} - \mathbf{x}_0| + t_0 - t \ge 0,$$

then $u(\mathbf{x}_0, t_0) = 0$*.*

The following result on L^2 multipliers is identical to Lemma 2.8.5 in (Stolk, 2000).

Lemma 2. *Let* (E, B, μ) *be a measure space,* $\{r_m\} \subset L^\infty(E, B, \mu)$ *with* $\|r_m\|_{L^\infty(E,B,\mu)} \le R \in \mathbf{R}_+$*,* $\{f_m\} \subset L^2(E, B, \mu)$ *with* $\|f_m\|_{L^2(\mu)} \le F \in \mathbf{R}_+$ *for all* $m \in \mathbf{N}$*. Suppose that* $r_m \to 0$ *in* μ*-measure. Then for any* $g \in L^2(E, B, \mu)$*,*

$$\lim_{m\to\infty} \int_E r_m f_m g d\mu = 0. \tag{49}$$

Proof. Suppose on the contrary that such sequences $\{r_m\}, \{f_m\}$ and square-integrable g exist, also an $\eta > 0$, for which the left-hand side of (49) remains $\ge \eta$ along a common subsequence. Without loss of generality, renumber the subsequence so that

$$\left| \int_E r_m f_m g d\mu \right| \ge \eta,\ m \in \mathbf{N}. \tag{50}$$

Convergence in measure of $\{r_m\}$ means that for any $\epsilon > 0$,

$$\mu[E_\epsilon(r_m)] \to 0 \text{ as } m \to \infty, \text{ where } E_\epsilon(r_m) = \{\mathbf{x} \in E : |r_m(\mathbf{x})| \geq \epsilon\}.$$

Choose ϵ so that $\epsilon F \|g\|_{L^2(E,B,\mu)} < \eta/2$.

From this definition and the Cauchy-Schwarz inequality, one sees that

$$\left| \int_E r_m f_m g d\mu \right| \leq \epsilon \int_{E \setminus E_\epsilon(r_m)} |f_m g| d\mu + R \int_{E_\epsilon(r_m)} |f_m g| d\mu$$

$$\leq \epsilon F \|g\|_{L^2(E,B,\mu)} + RF \left(\int_{E_\epsilon(r_m)} g^2 d\mu \right)^{\frac{1}{2}} < \frac{\eta}{2} + RF \left(\int_{E_\epsilon(r_m)} g^2 d\mu \right)^{\frac{1}{2}}$$

By passing if necessary to a further subsequence, we may assume that

$$\mu[E_\epsilon(r_m)] \leq 2^{-m} \ \Rightarrow \ \sum_m \mu[E_\epsilon(r_m)] < \infty.$$

Thus the characteristic functions of the sets $E_\epsilon(r_m)$ are almost everywhere convergent to zero as $m \to \infty$. Since $|g|^2 \in L^1(E, B, \mu)$, it follows from the Lebesgue Dominated Convergence Theorem that for large enough m,

$$\left(\int_{E_\epsilon(r_m)} g^2 d\mu \right)^{\frac{1}{2}} < \frac{\eta}{2RF}$$

Thus the left-hand side of (50) can be made smaller than η, a contradiction. $\square$

Lemma 3. *Suppose that $\{a_m\}_{m=1}^\infty \subset L^\infty(\mathbf{R}^n)$ converges in measure to $a \in L^\infty(\mathbf{R}^n)$, and that $A_m, m \in \mathbf{N}$ and $A \in \mathcal{B}(L^2(\mathbf{R}^n))$ are defined by*

$$(A_m u)(\mathbf{x}) = a_m(\mathbf{x}) u(\mathbf{x}), \ (Au)(\mathbf{x}) = a(\mathbf{x}) u(\mathbf{x}), \ u \in L^2(\mathbf{R}^n).$$

Then $A_m \to A$ strongly. The same is true for similar sequences of operators on $L^2(\mathbf{R}^n)^p$ defined by sequences of $p \times p$ matrix-valued functions whose components converge in measure.

Proof. In fact, the operators so defined are self-adjoint, and

$$\|A_m u - Au\|^2 = \int (a_m - a)[(a_m - a)u]u \to 0,$$

as follows from Lemma 2, taking $a_m - a$ for r_m, $(a_m - a)u$ for f_m, and u for g in the notation of that lemma. $\square$

Theorem 8. *In the setting of Theorem 7, suppose that the matrix-valued functions A, B, and Q are bounded and measureable, rather than smooth. Suppose that $\epsilon > 0$, and that $\tau \in \mathbf{R}_+$ satisfies*

$$\tau A(\mathbf{x}) + \sum_{i=1}^{n} K_i \xi_i \geq \epsilon \tag{51}$$

for almost every $\mathbf{x} \in \mathbf{R}^n$ and every $\xi \in \mathbf{R}^n$ for which $|\xi| = 1$. Suppose further that $\Omega \subset \mathbf{R}^{n+1}$ is bounded and open, and that $f(\mathbf{x}, t) = 0$ if $\tau|\mathbf{x} - \mathbf{x}_0| + t_0 - t \geq 0$ for every $(\mathbf{x}_0, t_0) \in \Omega$. Then the causal weak solution u of (2) *vanishes in Ω.*

Proof. According to the Lebesgue Differentiation Theorem, there are continuous

$$\bar{A}_m, \bar{B}_m \in C^0(\mathbf{R}^n)^{p\times p}, \bar{Q}_m \in C^0(\mathbf{R}^{n+1})^{p\times p} \bigcap L^1(\mathbf{R}, L^\infty(\mathbf{R}^3)^{p\times p})$$

so that $\bar{A}_m(\mathbf{x}) \to A(\mathbf{x})$ for almost all $\mathbf{x} \in \mathbf{R}^3$, and similarly for B and Q. Furthermore, these sequences are uniformly bounded. Standard smoothing results show that we may replace C^0 by C^∞, obtaining sequences $A_m, B_m \in C^\infty(\mathbf{R}^n)^{p\times p}, Q_m \in C^\infty(\mathbf{R}^{n+1})^{p\times p} \bigcap L^1(\mathbf{R}, L^\infty(\mathbf{R}^3)^{p\times p})$ converging to A, B, and Q almost everywhere and uniformly bounded. It follows that these sequences converge in measure as well. From Lemma 3, the corresponding operators converge strongly, whence the sequence of causal weak solutions $\{u_m\}$, obtained by replacing A with A_m and so on, converges to u in $L^2_{\mathrm{loc}}(\mathbf{R}, L^2(\mathbf{R})^p)$, according to Theorem 4. Since (51) holds almost everywhere, it follows that A_m satisfies the spectral inequality (48) in $\mathbf{R}^n$ for sufficiently large m, whence Corollary 11 implies that u_m vanishes in Ω. Since Ω is bounded, $u_m \to u$ in $L^2(\bar{\Omega})$, whence the conclusion follows. □

Corollary 12. *Denote by c_p the maximum quasi-p-wave velocity of the viscoelastic system* (44)*, defined as*

$$c_p = \text{ess sup}\{\lambda_{\max}(\Gamma^e(\mathbf{x})[\xi\xi^T]/\rho(\mathbf{x}) : \mathbf{x}, \xi \in \mathbf{R}^3, \xi^T\xi = 1\}.$$

Suppose that $(\mathbf{x}_0, t_0)$ satisfies

$$|\mathbf{x} - \mathbf{x}_0| > c_p(t_0 - t)$$

for every $(\mathbf{x}, t) \in \operatorname{supp} \mathbf{f}$. Then the causal weak solution $(\sigma, \mathbf{v})$ of (44) *vanishes in a neighborhood of $(\mathbf{x}_0, t_0)$.*

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