Solutions to Integro-Differential Evolution Equations with Discontinuous Coefficients

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Personal Background

- Current: TRIP-VIGRE Postdoc in CAAM dept. at Rice University
  Advisor: W. W. Symes
- PhD in mathematics from the University of Washington in 2006
  Advisor: K. P. Bube
  The One-Dimensional Seismic Inverse Problem on a Viscoacoustic Medium
- MS in mathematics from the University of Washington in 2003
  Advisor: E. L. Stout
- BS in mathematics from New Mexico Tech in 2000
  Advisor: D. R. Arterburn

Topics

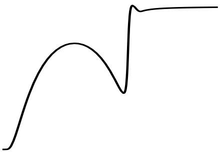
- The limitations of linearization
- The abstract forward model
  - Work based on the second-order equation of Lions-Magenes [Lions and Magenes, 1972, Non-homogeneous boundary value problems and applications, vol. 1]
- Continuous dependence on coefficients
  - Second-order equation proven by Stolk [Stolk, 2000, On the Modeling and Inversion of Seismic Data]
- The future: using the abstract forward model for general inversion, starting with 1-D

The trouble with linearization

Linearizing around a smooth background

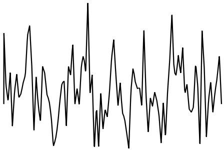
Most techniques used to solve inverse problems assume the functions describing the medium are oscillatory perturbations around a smooth medium.

$$c = c_0 + \delta c$$



Smooth Background

c_0



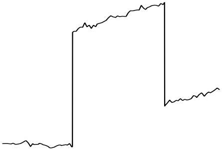
Rough Perturbation

δc

Combined Function

c

Linearization depends on the separation of the medium into a low frequency (smooth) component and a high frequency (oscillatory) component. This does not match with reality, where there is no separation of scales.



Information at all frequencies

- The good news: there are projects currently underway in numerical nonlinear inversion (Dong Sun)
- The bad news: there is no theoretical framework justifying the ability to invert for nonsmooth media
- Even worse news: not even in one dimension
  - The H^1 theory and the work of Bube on discontinuous media is insufficient for general discontinuous media.

The ultimate goal: full nonlinear inversion

This project is the first step towards full nonlinear inversion.

- Analysis of the forward problem for general coefficients (L^∞)
  - Existence proof = convergence of finite element method
  - Continuity of solutions w.r.t. coefficients
- The model is abstract enough that it covers many seismic models
  - Acoustics
  - Elastics
  - Viscoelastics

The Abstract Form of the Differential Equation

We consider the differential equation

$$Au' + Du + Bu + R[u] \quad = \quad f \in L^2(\mathbb{R}, H)$$

H is a Hilbert space (like L^2, functions with finite energy)

- $A \in \mathcal{B}(H)$ is self-adjoint and positive-definite
- D skew-adjoint with dense domain $V \in H$
- $B \in \mathcal{B}(H)$
- $R[u](t) = \int Q(t-s)u(s)\,ds$, where $Q \in C(\mathbb{R}, \mathcal{B}(H))$ and $Q(t) = 0$ for $t < 0$

Example: Acoustics in the abstract framework

Consider the acoustic wave equation on a domain $\Omega \in \mathbb{R}^3$

$$\begin{aligned} \rho \frac{\partial \mathbf{v}}{\partial t} &= -\nabla p + \mathbf{f}, \\ \frac{1}{\kappa}\frac{\partial p}{\partial t} &= -\nabla \cdot \mathbf{v}. \end{aligned}$$

Define $H = (L^2(\Omega))^4$. Then $u = (p, v_1, v_2, v_3)^T$,

$$A = \begin{pmatrix} \frac{1}{\kappa} & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 & 0 \end{pmatrix}$$

Example: Viscoelasticity

Under assumptions of linearity and causality, the viscoelastic wave equation is

$$\rho \frac{\partial v_i}{\partial t} = \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} + f_i$$

$$\frac{\partial \sigma_{kl}}{\partial t} = \sum_{i,j} C_{ijkl} *_t \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

One choice of C is $C(\mathbf{x},t) = \widetilde{C}(\mathbf{x})(\delta(t) - a(\mathbf{x})e^{-\alpha(\mathbf{x})t}H(t))$. δ is the Dirac delta, H is the Heaviside function, and $*_t$ denotes convolution in time.

Example: Viscoelasticity

Moving the convolution from the spatial derivatives to the time derivatives, we get

$$\begin{aligned} f_i &= \rho(\mathbf{x})\frac{\partial v_i}{\partial t} - \sum_j \frac{\partial \sigma_{ij}}{\partial x_j} \\ 0 &= \sum_{i,j} \widehat{C}_{ijkl}(\mathbf{x})\frac{\partial \sigma_{ij}}{\partial t} - \frac{1}{2}\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_l}{\partial x_k}\right) \\ &\quad + \sum_{i,j} b_{ijkl}(\mathbf{x})\sigma_{ij} + q_{ijkl}(\mathbf{x},t) *_t \sigma_{ij}. \end{aligned}$$

which fits with into the abstract model in a natural way

$$Au' + Du + Bu + R[u] = f$$

What Coefficients Correspond to Bounded Operators?

Our standard examples have $H = L^2$.

If a function is in L^∞, then the operator on L^2 given by multiplication against that function is a bounded operator.

$$\|fg\|_2 \leq \|f\|_\infty \|g\|_2$$

So the abstract equation covers the case where we are trying to solve a differential equation with L^∞ coefficients for an L^2 solution.

Existence and uniqueness

Theorem

A unique causal solution to the differential equation

$$Au' + Du + Bu + R[u] = f$$

exists provided that $f \in L^2(\mathbb{R}, H)$ *is causal:* $\operatorname{supp} f \subset [T_0, \infty)$ *for some* $T_0 \in \mathbb{R}$*, and that the causal convolution kernel* $Q \in L^1(\mathbb{R}, \mathcal{B}(H))$ *is continuous in* $\mathbb{R}_+$*:* $Q \in C^0(\mathbb{R}_+, \mathcal{B}(H))$*. The solution* $u \in L^2(\mathbb{R}, H)$ *and* $\operatorname{supp} u \subset [T_0, \infty)$*.*

Let $\{w_k\}_{k=1}^{\infty} \subset V$ form a basis for H in V. Define the functions

$$u_m(t) = \sum_{k=1}^{m} g_{km}(t) w_k,$$

where the g_{km}'s are determined by the differential equation

$$\begin{aligned} \langle u_m'(t), Aw_l\rangle - \langle u_m(t), Dw_l\rangle & & \\ + \langle u_m(t), B^* w_l\rangle + \langle u_m(t), R^*[w_l](t)\rangle &= \langle f(t), w_l\rangle, \qquad 1 \le l \le m, \\ u_m &= 0 \text{ for } t \le T_0. \end{aligned}$$

Convergence of the Finite Element Method

Convergence of the finite element approximations follow from energy estimates.

For the physical models, the abstract energy used here is the physical energy of the system.

If we have a sequence of equations

$$A_m u_m' + D_m u_m + B_m u_m + R_m[u_m] = f$$

and the coefficients converge in the weak sense

$$\lim_{m\to\infty} \|(A_m - A)w\| \to 0 \text{ for all } w \in H$$

Then u_m converges in measure.

If $H = L^2(\mathbb{R}^n)$ and the coefficients are L^∞, then L^1 convergence of the coefficients gives strong convergence of the solutions.

We can also take the derivative of the solution to the differential equation with respect to the coefficients.

If u_h is the solution to the differential equation with coefficients

$$A_h = A + h\delta A,\ B_h = B + h\delta B,\ Q_h = Q + h\delta Q$$

Then $(u_h - u)/h$ converges to the directional derivative of u in L^2.

With existence, uniqueness, and convergence with respect to coefficients out of the way, the next step is to head towards nonlinear inversion of the 1-D problem.

What do we know so far?

- There is a one-to-one correspondence between H^1 impedances and L^2 impulse responses h which satisfy the acoustic transparency property

$$\langle f, h * f \rangle \geq \epsilon \|f\|^2$$

- Impedances which are functions of bounded variation satisfy acoustic transparency
- There exist non-BV functions which are not transparent

Conjecture

Based on what we know, we make the following conjecture:

- The natural realm of inversion for one-dimension is bounded variation
  - If the impedance is not BV, then transparency will fail
  - If the impedance is BV, then nonlinear inversion is possible

We hope to approach these problems using the convergence results for the abstract problem.

Conclusion

- We have shown that first-order integro-differential equations with coefficients forming bounded operators on Hilbert spaces have unique solutions in an appropriate sense.
- These equations include the acoustic wave equation, the elastic wave equation, and the viscoelastic wave equation with discontinuous coefficients as special cases.
- These solutions are continuous with respect to all parameters, that is, the coefficients of the equation, the initial condition, and the forcing function.
- We hope to use these results to establish nonlinear inversion for the one-dimensional problem

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