Operator Upscaling
for the Wave Equation

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January 26, 2007

Outline

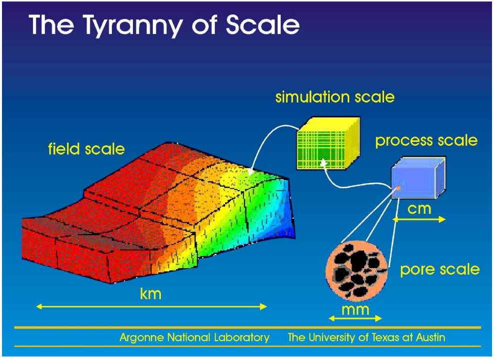
- Upscaling in the Context of Multiscale Methods.
- Upscaling for the Acoustic Wave Equation
    - Description of the Method
    - Numerical Implementation
    - Numerical Experiments
- Work in progress: Upscaling for the Elastic Wave Equation

Multiscale Methods

- Why do we need multiscale methods?
  - Many processes in nature involve multiple scales.
- **Goal:** to design a numerical technique that
  - produces accurate solution on the coarse scale;
  - is more efficient than solving full fine scale problem.

Multiscale problems:

- composite materials (10^{-9} m - large scales depend on applications),
- protein folding (10^{-15} - 10^{-1} s),
- flow in porous media (10^{-2} - 10^{4} m).



http://www.ticam.utexas.edu/Groups/SubSurfMod/ACTI/IPARS.htm

Upscaling Methods

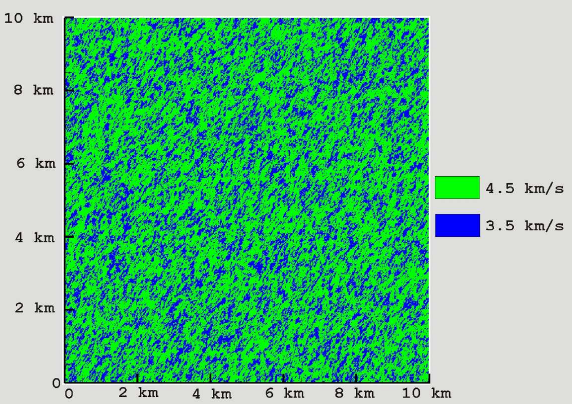
Highly detailed physical models $\xrightarrow{\textbf{Upscaling}}$ Feasible simulation grids

Upscaling is the process of converting the problem from the fine scale where physical parameters are defined to a coarse scale.

- Averaging: Review by Renard and Marsily (1997).
- Renormalization: King (1989).
- Homogenization: Bensoussan, Lions, Papanicolaou (1978).
- Multiscale FEM: Hou, Wu (1997).
- Mortar Upscaling: Peszynska, Wheeler, Yotov (2002).
- Variational Multiscale Method: Hughes (1995).
- Operator Upscaling: Arbogast, Minkoff, Keenan (1998).

Velocity Model

- Mechanical properties of the Earth are very heterogeneous.



- **Fine scale:** ≈ 10 m. **Large scale:** $\approx 10^4 - 10^5$ m.
- **Typical grid size:** $10^6 - 10^8$ in 2D, $10^9 - 10^{12}$ in 3D.

Model problem: The Acoustic Wave Equation

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = f$$

p is the pressure,
$\mathbf{u}$ is the acceleration,

f is the source of acoustic energy,
$c(x, y)$ is the sound velocity.

The First Order System

$$\mathbf{u} = -\nabla p \text{ in } \Omega,$$
$$\frac{1}{c^2(x,y)}\frac{\partial^2 p}{\partial t^2} + \nabla \cdot \mathbf{u} = f \text{ in } \Omega$$

Boundary and Initial Conditions

$$\mathbf{u} \cdot \nu = 0, \text{ on } \partial\Omega,$$
$$p = 0, \text{ on } \partial\Omega,$$

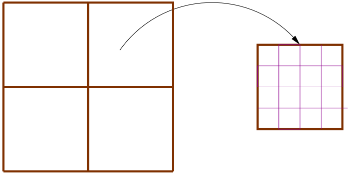
$$p(0, x, y) = p_0,$$

$$\frac{\partial p}{\partial t}(0, x, y) = p_1.$$

Finite Element Spaces

Goal: Capture fine-scale behavior on the coarse grid.

Two-scale grid:



Fine scale: Raviart-Thomas (RT-0) spaces on each coarse element:

- Pressure: W_h={piecewise discontinuous constant functions}
- Acceleration:
$$\delta \mathbf{V}_h = \{\delta \mathbf{v} = (a_1 x + b_1, a_2 y + b_2) : \nabla \cdot \delta \mathbf{v} \in \mathbf{L}^2(E_c), \\ \delta \mathbf{v} \cdot \boldsymbol{\nu} = 0, \text{ on } \partial E_c\}$$

Coarse scale:

$$V_H = \{\mathbf{v} = (a_1 x + b_1, a_2 y + b_2) : \nabla \cdot \mathbf{v} \in \mathbf{L}^2, \mathbf{v} \cdot \boldsymbol{\nu} = 0, \text{ on } \partial\Omega\}$$

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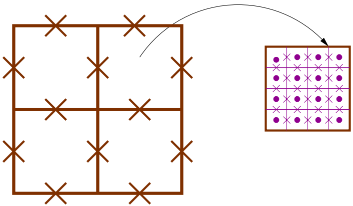

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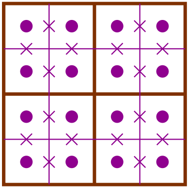
Two-Stage Algorithm: $\mathbf{V}_{H,h} = \mathbf{V}_H \oplus \delta\mathbf{V}_h$

Step 1: On each coarse element E_c solve the subgrid problem:



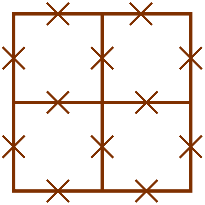
Find $(\delta\mathbf{U}, P) \in \delta\mathbf{V}_h(E_c) \times W(E_c)$ such that:

$$\left(\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2}, w\right)_{E_c} - (\nabla \cdot (\delta\mathbf{U} + \mathbf{U}_H), w)_{E_c} = (f, w)_{E_c},$$

$$(\delta\mathbf{U} + \mathbf{U}_H, \delta\mathbf{v})_{E_c} - (P, \nabla \cdot \delta\mathbf{v})_{E_c} = 0,$$

for all $\delta\mathbf{v} \in \delta\mathbf{V}_h(E_c)$ and $w \in W(E_c)$.

Step 2: Use the subgrid solutions to solve the coarse-grid problem:



Find $\mathbf{U}_H \in \mathbf{V}_H$ such that:

$$((\mathbf{U}_H + \delta\mathbf{U}), \mathbf{v})_\Omega - (P, \nabla \cdot \mathbf{v})_\Omega = 0,$$

for all $\mathbf{v} \in \mathbf{V}_H$.

Parallel Performance

$$\frac{\text{Cost of subgrid problems}}{p} + \text{Cost of coarse problem}$$

- Subgrid problems: Embarrassingly parallel
  - No communication between processors.
  - No additional ghost-cell memory allocations.
  - Explicit difference scheme.
  - Later implementation avoids numerical Green's functions.
- Coarse problem: Solve in serial. Explicit difference equation.

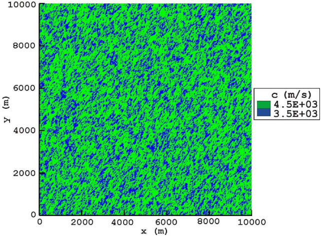
p	total time (FD)	total time (NG)	total time (noNG)	subgrid problems	coarse problem	post process
1	29.43	45.65	29.70	29.69	0.00060	0.0026
2	-	23.18	15.46	15.38	0.00045	0.0711
4	-	11.72	7.63	7.56	0.00049	0.0707
6	-	7.97	5.23	5.14	0.00048	0.0749
8	-	7.05	4.37	4.26	0.00046	0.0896
12	-	4.92	3.07	2.94	0.00045	0.1150

- FD – finite differences
- NG – numerical Green's functions
- noNG – no numerical Green's functions

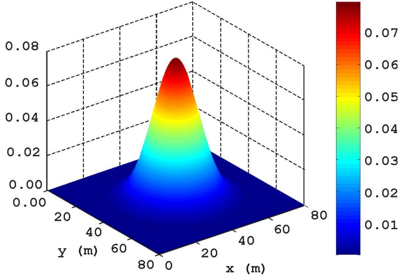
Acoustic Numerical Experiment

- Domain is of size 10×10 km.
- Fine grid: 1000×1000. Coarse grid: 100×100.
- Gaussian source, 350 time steps.
- Mixture of two materials with sound velocities of 3500 and 4500 m/s.

Sound velocity field

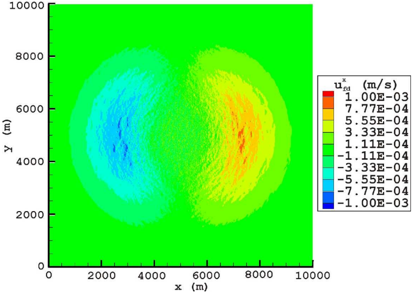


Gaussian Source

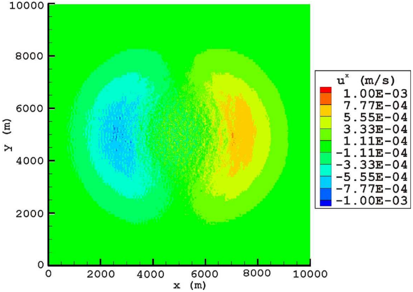


Horizontal Acceleration

Full finite-difference solution

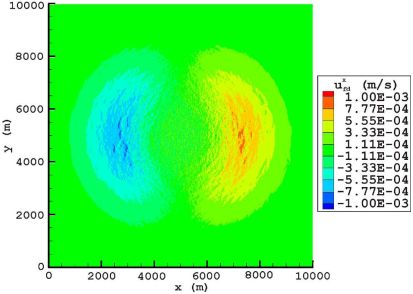


Augmented upscaled solution

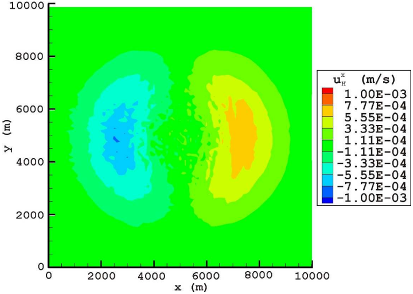


Horizontal Acceleration

Full finite-difference solution



Coarse component of solution



Vdovina, Minkoff, Korostyshevskaya (2005)

Work in Progress: Elastic Wave Equation

The First Order System

$$\rho(\mathbf{x})\frac{\partial \mathbf{v}(t,\mathbf{x})}{\partial t} = \nabla\cdot\boldsymbol{\sigma} + \mathbf{f},$$
$$\rho(\mathbf{x})\frac{\partial \mathbf{u}(t,\mathbf{x})}{\partial t} = \rho(\mathbf{x})\mathbf{v}(t,\mathbf{x}),$$

$\mathbf{v}$ is velocity,
$\mathbf{u}$ is displacement,
ρ is density,

$\boldsymbol{\sigma}$ is the stress tensor,
$\mathbf{f}$ is a body force,
$\mathbf{x}$ is in $\mathbb{R}^3$

Boundary and Initial Conditions

$$\mathbf{u}(0,\mathbf{x}) = \mathbf{u}_0(\mathbf{x}),$$
$$\mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}),$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \Gamma.$$

Weak Formulation

- Cohen (2002), Komatitch *et.al.*(1999), SpecFEM 3D
- Velocity and displacement space

$$\mathbf{W} = \{\mathbf{w} \in \mathbf{H}^1(\Omega),\, \mathbf{w}(\mathbf{x}) = \mathbf{0} \text{ on } \Gamma\}.$$

Find $\mathbf{v}(t,\mathbf{x})$ and $\mathbf{u}(t,\mathbf{x})$ in $\mathbf{W}$ such that:

$$\begin{aligned} \left(\rho \frac{\partial \mathbf{v}}{\partial t}, \mathbf{w}\right) &= -\left(\boldsymbol{\sigma}, \nabla \mathbf{w}\right) + \left(\mathbf{f}, \mathbf{w}\right), \\ \left(\rho \frac{\partial \mathbf{u}}{\partial t}, \mathbf{w}\right) &= \left(\rho \mathbf{v}, \mathbf{w}\right), \end{aligned}$$

for $\mathbf{w}(\mathbf{x})$ in $\mathbf{W}$ and $t \in [0, T]$.

- Eliminate components of the stress tensor:

$$\sigma_{i,j} = \lambda \sum_k^3 \frac{\partial u_k}{\partial x_k} \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Weak Formulation (continued)

- First component of velocity:

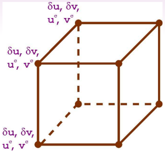
$$\begin{aligned}\left(\rho\frac{\partial v_1}{\partial t}, w\right) &= -\left((\lambda+2\mu)\frac{\partial u_1}{\partial x}+\lambda\frac{\partial u_2}{\partial y}+\lambda\frac{\partial u_3}{\partial z}, \frac{\partial w}{\partial x}\right)\\ &- \left(\mu\frac{\partial u_1}{\partial y}+\mu\frac{\partial u_2}{\partial x}, \frac{\partial w}{\partial y}\right)\\ &- \left(\mu\frac{\partial u_1}{\partial z}+\mu\frac{\partial u_3}{\partial x}, \frac{\partial w}{\partial z}\right) + (f_1, w)\,,\end{aligned}$$

- First component of displacement:

$$\left(\rho\frac{\partial u_1}{\partial t}, w\right) = (\rho v_1, w)\,,$$

- Upscale both variables

Finite Element Method



- Subgrid Scale:
  - piece-wise trilinear functions
  - zero boundary conditions
- Coarse Scale:
  - piece-wise trilinear functions
  - original boundary conditions

- First component of velocity:

$$
\begin{aligned}
&\left(\rho \frac{\partial}{\partial t}(v_1^c + \delta v_1), w\right) \\
&= -\left((\lambda + 2\mu)\frac{\partial}{\partial x}(u_1^c + \delta u_1) + \lambda \frac{\partial}{\partial y}(u_2^c + \delta u_2) + \lambda \frac{\partial}{\partial z}(u_3^c + \delta u_3), \frac{\partial}{\partial x} w\right) \\
&- \left(\mu \frac{\partial}{\partial y}(u_1^c + \delta u_1) + \mu \frac{\partial}{\partial x}(u_2^c + \delta u_2), \frac{\partial w}{\partial y}\right) \\
&- \left(\mu \frac{\partial}{\partial z}(u_1^c + \delta u_1) + \mu \frac{\partial}{\partial x}(u_3^c + \delta u_3), \frac{\partial w}{\partial z}\right) + (f_1, w)\,,
\end{aligned}
$$

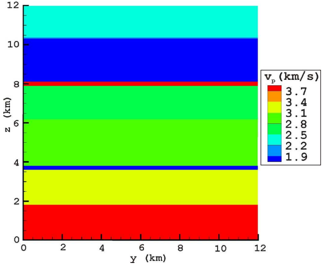
- First component of displacement:

$$
\left(\rho \frac{\partial}{\partial t}(u_1^c + \delta u_1), w\right) = (\rho(v_1^c + \delta v_1), w)\,,
$$

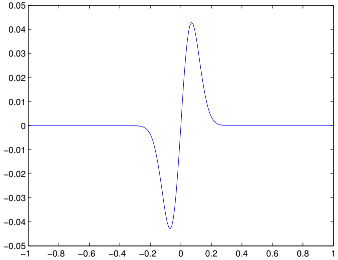
Elastic Numerical Experiment I

- Domain is of size $12 \times 12 \times 12$ km.
- Fine grid: $120 \times 120 \times 120$. Coarse grid: $24 \times 24 \times 24$.
- Gaussian source, 35 time steps.
- Layered medium.

Compressional velocity (*yz*-slice)

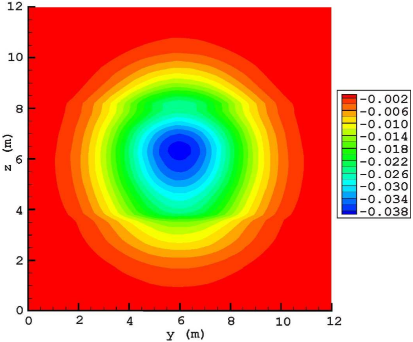


Source in 1D

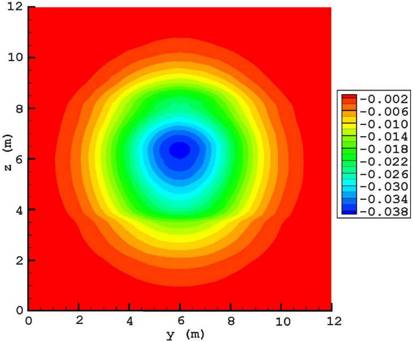


First Component of the Velocity Solution (yz-plane)

Full finite-element solution

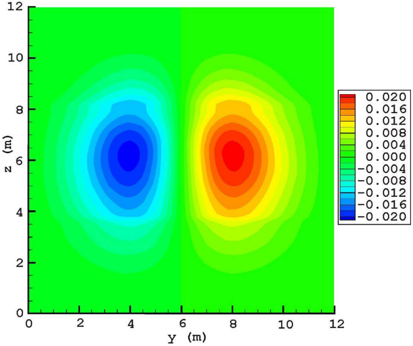


Reconstructed upscaled solution

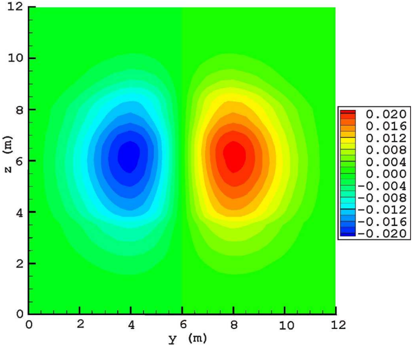


Second Component of the Velocity Solution (*yz*-plane)

Full finite-element solution

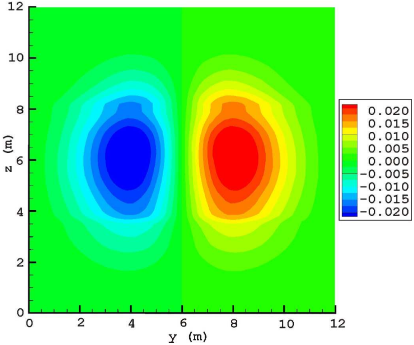


Reconstructed augmented solution

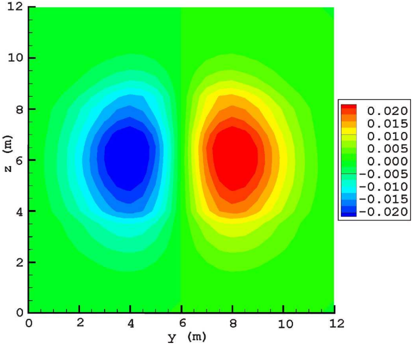


Second Component of the Velocity Solution (yz-plane)

Full finite-element solution



Coarse component of solution



Summary and Future Work

- Operator upscaling captures local phenomena on the coarse scale.
- Elastic equation: extension of operator upscaling to the mixed formulation.
  - perfectly matched layers
  - higher order interpolating polynomials
- Operator upscaling with discontinuous Galerkin methods for wave equations.

Acknowledgment

- This research was performed with funding from the Collaborative Math-Geoscience Program at NSF (2002-2005) and fellowship provided by NASA Goddard's Earth Sciences and Technology Center (GEST) (2005-2006).