Fast sweeping methods and applications to traveltime tomography

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Outline

- Eikonal equations.
- Fast sweeping methods for eikonal equations: direct problems.
- Traveltime tomography methods: inverse problems.
- Outlook and future works.

Eikonal equations

- Eikonal eqn:

$$|\nabla T(\mathbf{x})| = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \setminus \Gamma,$$
$$T(\mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \in \Gamma \subset \Omega,$$

- When $f \equiv 1$, $\Gamma = \{\mathbf{0}\}$ and $\tau(\mathbf{0}) = 0$ are given, the solution is the distance function:

$$\tau(\mathbf{x}) = |\mathbf{x}| = \sqrt{\mathbf{x}^T\mathbf{x}}.$$

Eikonal equations: cont.

- A nonlinear first-order partial differential equation;
- Theory: local existence of smooth solutions; generalized solutions may not be unique;
- Theory: the viscosity solution as a generalized solution is unique!!
- Applications: computer vision, medical imaging, robotic navigation, oil exploration, $\cdots$
- Mission: $O(M)$ algorithms to compute such viscosity solutions, where M is the number of unknown mesh points. What does this mean?

Eikonal eqns: numerics

- Dijkstra method for shortest distances (1959): inconsistent but unconditionally stable;
- Viscosity solutions and consistent monotone schemes (Crandall-Lions'83,'84);
- Typical methods on CARTESIAN meshes:
  - Upwinding: Vidale'88, van Trier-Symes'91, Kim-Cook'99, Qian-Symes'02: $O(M)$;
  - Jacobi iterations: Rouy-Tourin'92: $O(M^2)$;
  - Fast marching methods (Tsitsiklis'95, Sethian'96): $O(MlogM)$ and uncond. stable;
  - Fast sweeping methods (Boue-Dupuis'99, Zhao'04): $O(M)$ and uncond. stable.

$|\nabla T| = f(\mathrm{x})$: essentials

- Hyperbolic type equations: looking for information in an upwind fashion;
- Viscosity solution: computable by consistent monotone schemes;
- Once discretized based on a monotone numerical Hamiltonian, a nonlinear system needs solving efficiently;
- Fast sweeping methods exactly designed to achieve the above purpose.

Fast sweeping: ideas

- $$\begin{cases} |\nabla T(\mathbf{x})| = f(\mathbf{x}), & \mathbf{x} \in \Omega \setminus \Gamma, \\ T(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma \subset \Omega, \end{cases}$$

  where $f(\mathbf{x}) > 0$, $\Omega \subset R^d$: a bounded domain.
- Seek viscosity solution $T(\mathbf{x}) \geq 0$;
- *A priori* partitioning all the unknown characteristics into a finite number of groups according to their directions;
- Order all the nodes systematically according to those directions;
- Update all the nodes according to those orderings: efficient local solvers and Gauss-Seidel strategy.

Fast sweeping: local solvers

- Use a Cartesian mesh to discretize Ω with grid size h and $T_{i,j}$: solution at $\mathbf{x}_{i,j}$
- Apply a Godunov upwind scheme in 2-D case:

$$[\max(T_{i,j} - T_{xm}, 0)]^2 + [\max(T_{i,j} - T_{ym}, 0)]^2 = f_{i,j}^2 h^2,$$

$$T_{i,j} = \begin{cases} \frac{1}{2}\left(T_{xm} + T_{ym} + \sqrt{2h^2 f_{i,j}^2 - (T_{xm} - T_{ym})^2}\right), \\ \qquad \text{if } |T_{xm} - T_{ym}| < h f_{i,j}; \\ \min(T_{xm}, T_{ym}) + h f_{i,j}, \quad \text{otherwise.} \end{cases}$$

$$T_{xm} = \min(T_{i-1,j}, T_{i+1,j}),\ T_{ym} = \min(T_{i,j-1}, T_{i,j+1}).$$

Fast sweeping: an algorithm

- Initialization: assign exact values or interpolated values at grid points whose distances to Γ are less than h; other nodes assigned a very large value.
- Gauss-Seidel iterations based on the local solver and four alternating sweeping orderings:
  (1) $i = 1 : I, j = 1 : J$; (2) $i = I : 1, j = 1 : J$;
  (3) $i = I : 1, j = J : 1$; (4) $i = 1 : I, j = J : 1$.
- Iteration stops if $\|T^{new} - T^{old}\|_{l^1} \leq \delta$, where δ is a given convergence threshold value.

Fast sweeping: an anatomy

- Use a Cartesian mesh for a domain: $[a, b] \times [c, d]$
- Partition all characteristics into: right- and left- going segments, and up- and down- going segments.
- Cover right- and left-going segments by sweeping vertical lines rightward and leftward:

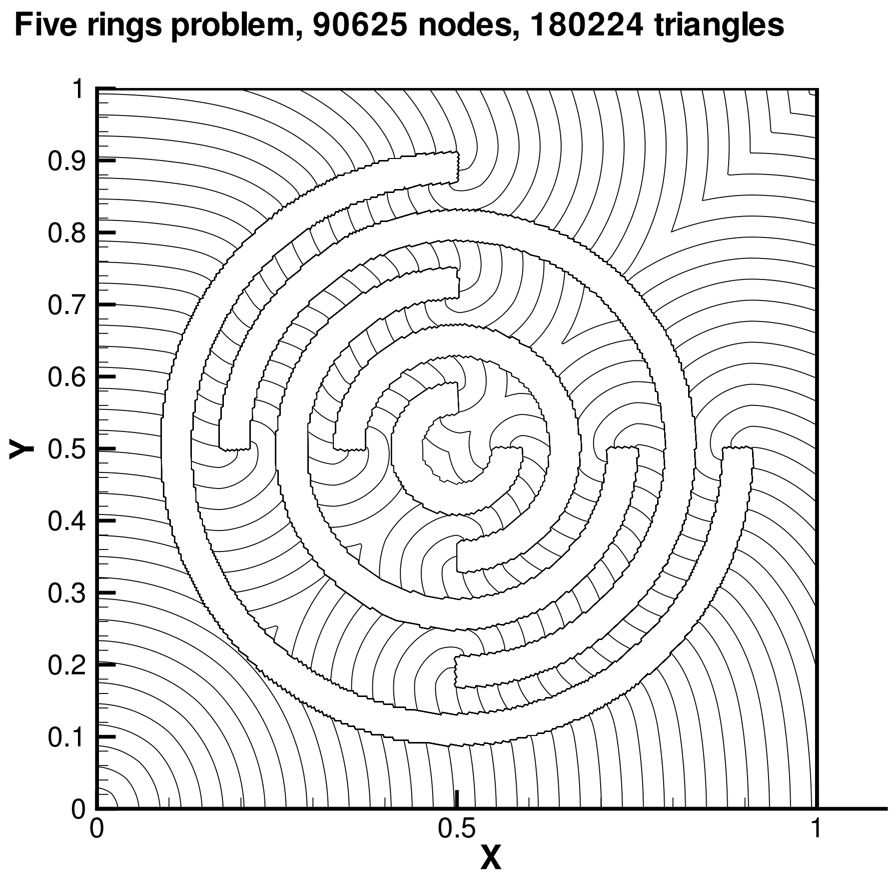
$$\{l_x : \{(x, y) : c \leq y \leq d\}, a \leq x \leq b\} \, ;$$
$$\{l_i : \{(x_i, y_j) : 1 \leq j \leq J\}, 1 \leq i \leq I\} \, ,$$

  which are naturally defined by the Cartesian mesh: easy to implement.
- Such a natural ordering no longer exists on a triangulated mesh. What to do?

Triangulation: novel orderings

- Question: how to sweep the unstructured nodes in a consecutive manner?
- Introduce multiple reference points and sort all the triangulated nodes according to their l^p metrics to each individual reference point (Qian-Zhang-Zhao, SIAM Numer. Analy., in press.)

Five rings



Five rings problem, 90625 nodes, 180224 triangles
Y
X
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1

Convergence order

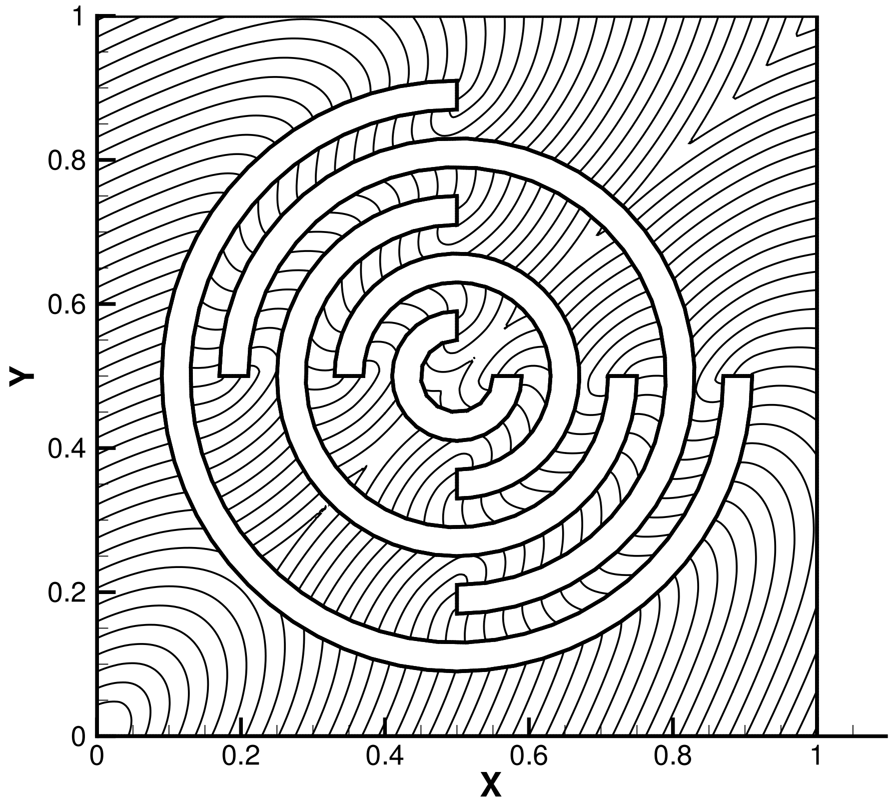
Table 1: Godunov numerical Hamiltonian.

	two-O		SFS-a	
Nodes	L^1	order	L^1	order
1473	7.71E-3	–	4.54E-2	–
5716	4.21E-3	0.87	2.54E-2	0.84
22785	2.18E-3	0.95	1.34E-2	0.92
90625	1.11E-3	0.97	6.90E-3	0.96

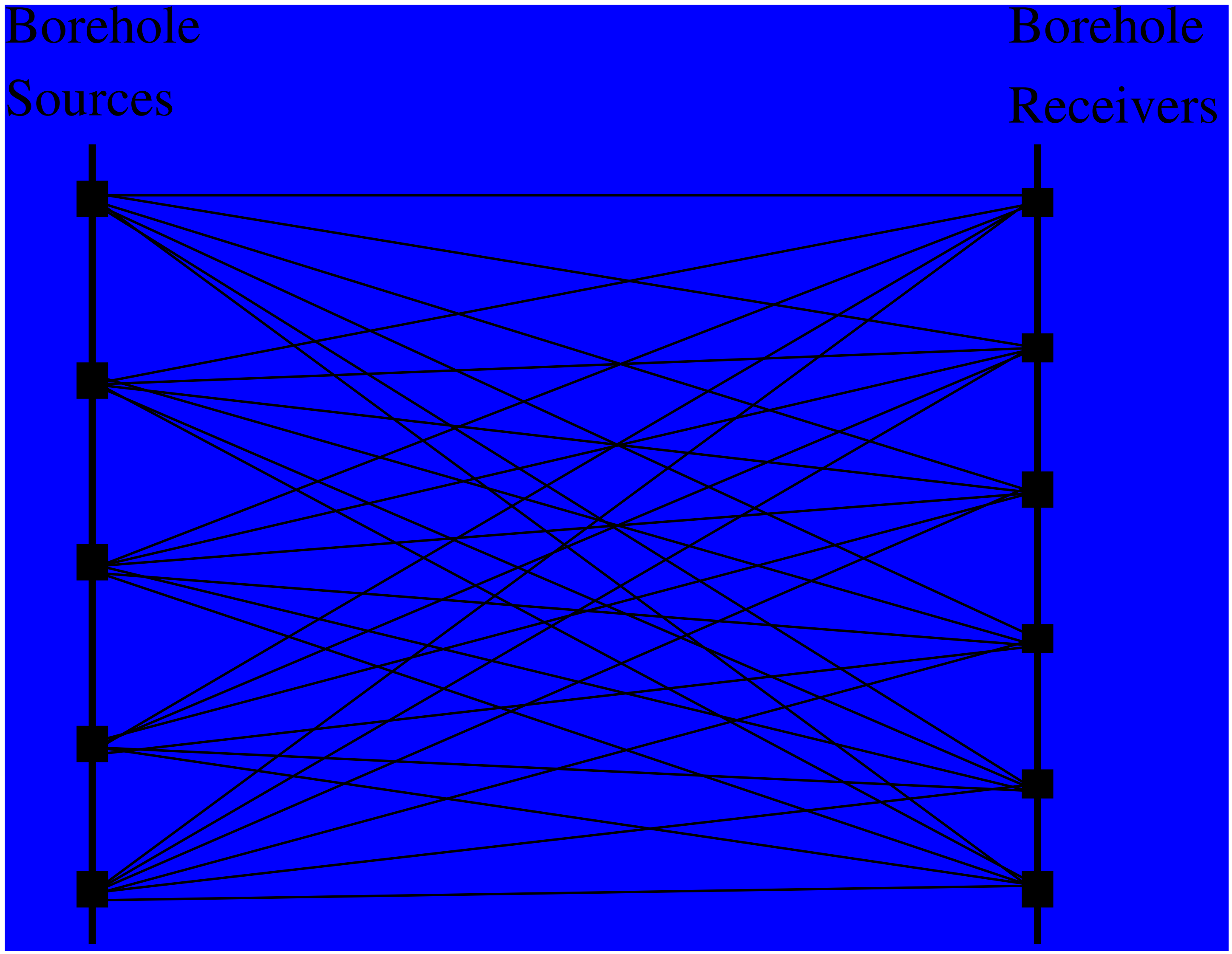
FSM for anisotropic media

- We have generalized the above approach to anisotropic media (Qian-Zhang-Zhao, J. Sci. Comp., to appear).

a=1, b=1, c=-0.5
63806 nodes, 31 iterations



Application: transmission traveltime tomography



Ray-tracing based tomography

- Traveltime between S and R: $t(S,R) = \int_S^R \frac{ds}{c}$.
- Fermat's principle serves as the foundation: First-Arrivals (FA) based.
- Both ray path and velocity (1/slowness) are unknown.
- Linearize the equation around a given background slowness with an unknown slowness perturbation.
- Discretize the interested region into pixels of constant velocities.
- Trace rays in the Lagrangian framework.
- Obtain a linear system linking slowness perturbation with traveltime perturbation.

Seismic traveltime tomography

- Transmission traveltime tomography estimates wave-speed distribution from acoustic, elastic or electromagnetic first-arrival (FA) traveltime data.
- Travel-time tomography shares some similarities with medical X-ray CT.
- Geophysical traveltime tomography uses travel-time data between source and receiver to invert for underground wave velocity.
- Seismic tomography usually is formulated as a minimization problem that produces a velocity model minimizing the difference between traveltimes generated by tracing rays through the model and those measured from the data: Lagrangian approaches.

Traveltime tomography

- We develop PDE-based Eulerian approaches to traveltime tomography to avoid ray-tracing.
- Traveltime tomography via eikonal eqns, adjoint state methods and fast eikonal solvers.
  - Sei-Symes'94, '95 formulated FA based traveltime tomography using paraxial eikonal eqns; they only illustrated the feasibility of computing the gradient by using the adjoint state method.
  - Our contribution: formulating the problem in terms of the full eikonal eqn, solving the eikonal eqn by fast sweeping methods and designing a new fast sweeping method for the adjoint eqn of the linearized eikonal eqn.

FA-based tomography: problem

- Traveltimes between a source S and receivers R on the boundary satisfy

$$c(\mathbf{x})|\nabla T| = 1, \quad T(\mathbf{x_s}) = 0.$$

- Forward problem: given $c > 0$, compute the viscosity solution based FAs from the source to receivers.
- Inverse problem: given both FA measurements on the boundary $\partial\Omega_p$ and the location of the point source $\mathbf{x}_s \in \partial\Omega_p$, invert for the velocity field $c(\mathbf{x})$ inside the domain Ω_p.

FA-based tomography: idea

- Forward problem: fast eikonal solvers; they are essential for inverse problems.
- Inverse problem: essential steps.
  - Minimize the mismatching functional between measured and simulated traveltimes.
  - Derive the gradient of the mismatching functional and apply an optimization method.
  - Linearize the eikonal eqn around a known slowness with an unknown slowness perturbation.
  - Solve the eikonal eqn for the viscosity solution: only FAs are used.

FA-based tomography: formulation

- The mismatching functional (energy),

$$E(c) = \frac{1}{2} \int_{\partial\Omega_p} |T - T^*|^2 ,$$

  where $T^*|_{\partial\Omega_p}$ is the data and $T|_{\partial\Omega_p}$ is the eikonal solution.

- Perturb c by $\epsilon\tilde{c} \Rightarrow$ Perturbation in T by $\epsilon\tilde{T}$ and in E by δE:

$$\delta E = \epsilon \int_{\partial\Omega_p} \tilde{T}(T - T^*) + O(\epsilon^2) .$$
$$T_x\tilde{T}_x + T_y\tilde{T}_y + T_z\tilde{T}_z = -\frac{\tilde{c}}{c^3} .$$

- Difficulty: δE depends on $\tilde{c}$ implicitly through $\tilde{T}$ and the linearized eikonal equation. Use the adjoint state method.

FA-based tomography: adjoint state

- Introduce λ satisfying

$$[(-T_x)\lambda]_x + [(-T_y)\lambda]_y + [(-T_z)\lambda]_z = 0\,,$$
$$(\mathbf{n}\cdot\nabla T)\lambda = T^* - T, \text{ on } \partial\Omega_p.$$

- Impose the BC to back-propagate the time residual into the computational domain.
- Simplify the energy perturbation further,

$$\frac{\delta E}{\epsilon} = \int_{\Omega_p} \frac{\tilde{c}\lambda}{c^3}\,.$$

- Choose $\tilde{c} = -\lambda/c^3 \Rightarrow$ Decrease the energy: $\delta E = -\epsilon\int_{\Omega_p}\tilde{c}^2 \leq 0$.

FA-based tomography: regularization

- Enforce
  1. $\tilde{c}|_{\partial\Omega_p} = 0$;
  2. $c^{k+1} = c^k + \epsilon\tilde{c}^k$ smooth.
- The first condition is reasonable as we know the velocity on the boundary.
- The second condition is a requirement on the smoothness of the update at each step.
- Regularize, $\nu \geq 0$,

$$\tilde{c} = -(I - \nu\Delta)^{-1}\left(\frac{\lambda}{c^3}\right),$$

$$\delta E = -\epsilon \int_{\Omega_p} (\tilde{c}^2 + \nu|\nabla\tilde{c}|^2) \leq 0\,.$$

FA-based tomography: multiple data sets (1)

- A single data set is associated with a single source.
- Incorporate multiple data sets associated with multiple sources into the formulation.
- Define a new energy for N sets of data:

$$E^N(c) = \frac{1}{2} \sum_{i=1}^{N} \int_{\partial \Omega_p} |T_i - T_i^*|^2,$$

where T_i are the solutions from the eikonal equation with the corresponding point source condition $T(\mathbf{x}_s^i) = 0$.

FA-based tomography: multiple data sets (2)

- Perturbation in the energy,

$$\frac{\delta E^N}{\epsilon} = \int_{\Omega_p} \frac{\tilde{c}}{c^3} \sum_{i=1}^{N} \lambda_i \,,$$

where λ_i is the adjoint state of T_i ($i = 1, \cdots, N$) satisfying

$$\{[-(T_i)_x]\lambda_i\}_x + \{[-(T_i)_y]\lambda_i\}_y + \{[-(T_i)_z]\lambda_i\}_z = 0 \,,$$
$$(\mathbf{n} \cdot \nabla T_i)\lambda_i = T_i^* - T_i \,.$$

- To minimize the energy $E^N(c)$, choose

$$\tilde{c} = -(I - \nu\Delta)^{-1} \left(\frac{1}{c^3} \sum^{N} \lambda_i \right) \,.$$

Fast sweeping for eikonal and adjoint equations

- Fast eikonal solvers: fast marching (Sethian, ...), fast sweeping (Zhao, Tsai, Cheng, Osher, Kao, Qian, Cecil, Zhang,...); see Engquist-Runborg'03 for more.
- The eikonal eqn is solved by the fast sweeping method (Zhao, Math. Comp'05).
- The adjoint equation for the adjoint state can be solved by fast sweeping methods as well.
- We have designed a new fast sweeping method for the adjoint eqn. (Leung-Qian'05)

Fast sweeping for the adjoint equation (1)

- Take the 2-D case to illustrate the idea:

$$(a\lambda)_x + (b\lambda)_z = 0\,,$$

  where a and b are given functions of (x, z).

- Consider a computational cell centered at (x_i, z_j) and discretize the equation in conservation form,

$$\begin{aligned} & \frac{1}{\Delta x}\left(a_{i+1/2,j}\lambda_{i+1/2,j} - a_{i-1/2,j}\lambda_{i-1/2,j}\right) \\ + \quad & \frac{1}{\Delta z}\left(b_{i,j+1/2}\lambda_{i,j+1/2} - b_{i,j-1/2}\lambda_{i,j-1/2}\right) = 0\,. \end{aligned}$$

Fast sweeping for the adjoint equation (2)

- λ on the interfaces, $\lambda_{i\pm 1/2,j}$ and $\lambda_{i,j\pm 1/2}$, determined by the propagation of characteristics, ie, upwinding,

$$
\begin{aligned}
& \frac{1}{\Delta x}\Big((a^{+}_{i+1/2,j}\lambda_{i,j} + a^{-}_{i+1/2,j}\lambda_{i+1,j})\Big) \\
- & \frac{1}{\Delta x}\Big((a^{+}_{i-1/2,j}\lambda_{i-1,j} + a^{-}_{i-1/2,j}\lambda_{i,j})\Big) \\
+ & \frac{1}{\Delta z}\Big((b^{+}_{i,j+1/2}\lambda_{i,j} - b^{-}_{i,j+1/2}\lambda_{i,j+1})\Big) \\
- & \frac{1}{\Delta z}\Big((b^{+}_{i,j+1/2}\lambda_{i,j-1} - b^{-}_{i,j+1/2}\lambda_{i,j})\Big) = 0\,,
\end{aligned}
$$

where $a^{\pm}_{i+1/2,j}$ denote the positive and negative parts of $a_{i+1/2,j}$.

Fast sweeping for the adjoint equation (3)

- Rewriting as

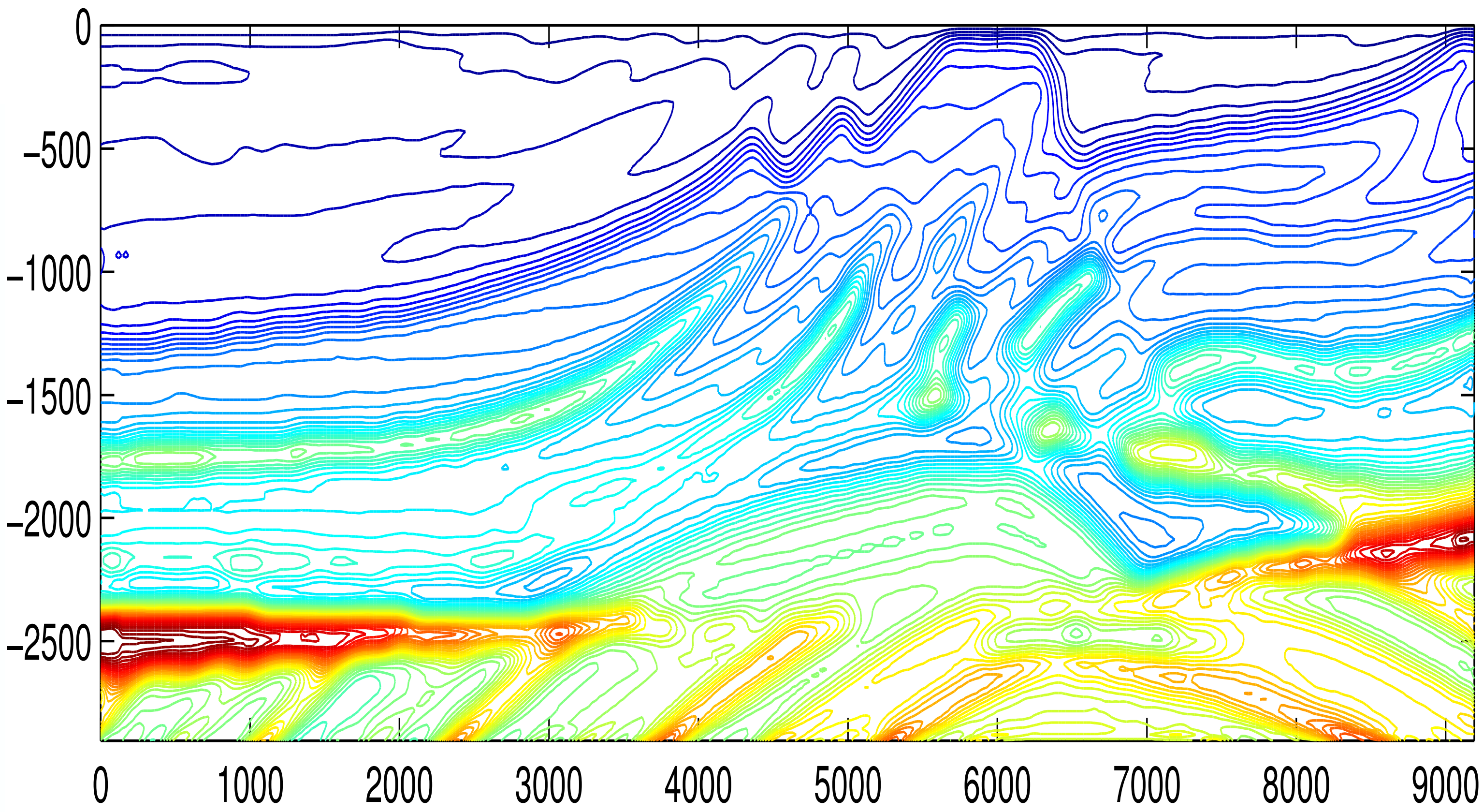
$$
\begin{aligned}
\alpha &= \left(\frac{a^{+}_{i+1/2,j} - a^{-}_{i-1/2,j}}{\Delta x} + \frac{b^{+}_{i,j+1/2} - b^{-}_{i,j-1/2}}{\Delta z} \right) \\
\alpha \lambda_{i,j} &= \frac{a^{+}_{i-1/2,j}\lambda_{i-1,j} - a^{-}_{i+1/2,j}\lambda_{i+1,j}}{\Delta x} \\
&\quad + \frac{b^{+}_{i,j-1/2}\lambda_{i,j-1} - b^{-}_{i,j+1/2}\lambda_{i,j+1}}{\Delta z}
\end{aligned}
$$

  which gives us an expression to construct a fast sweeping type method.
- Alternate sweeping strategy applies.

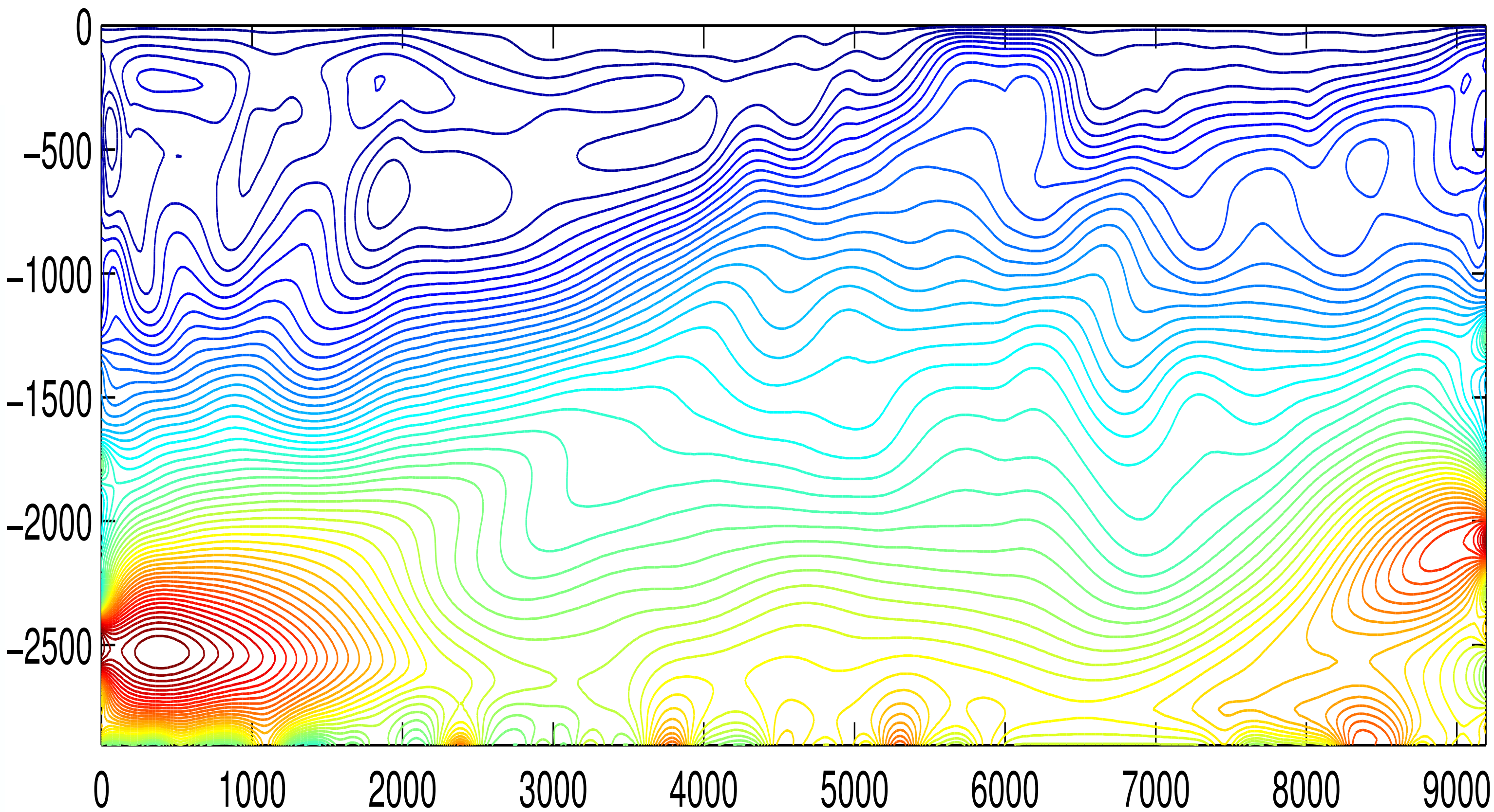
Other details

- The Poisson eqn is solved by FFT.
- The gradient descent method needs too many iterations.
- Use the limited memory Broyden, Fletcher, Goldfarb, Shanno (L-BFGS) method: a quasi-Newton optimization method (Byrd, Lu, Nocedal and Zhu'95).
- Ideal illuminations are assumed.

Marmousi: true model



Marmousi: 10 sources



Outlook and future works

- Fast sweeping methods are powerful for solving Hamilton-Jacobi equations;
- Many possible applications of these methods;
- Future works
  - Open to your suggestions ...