Traveltime computation and tomography based on the Liouville equation

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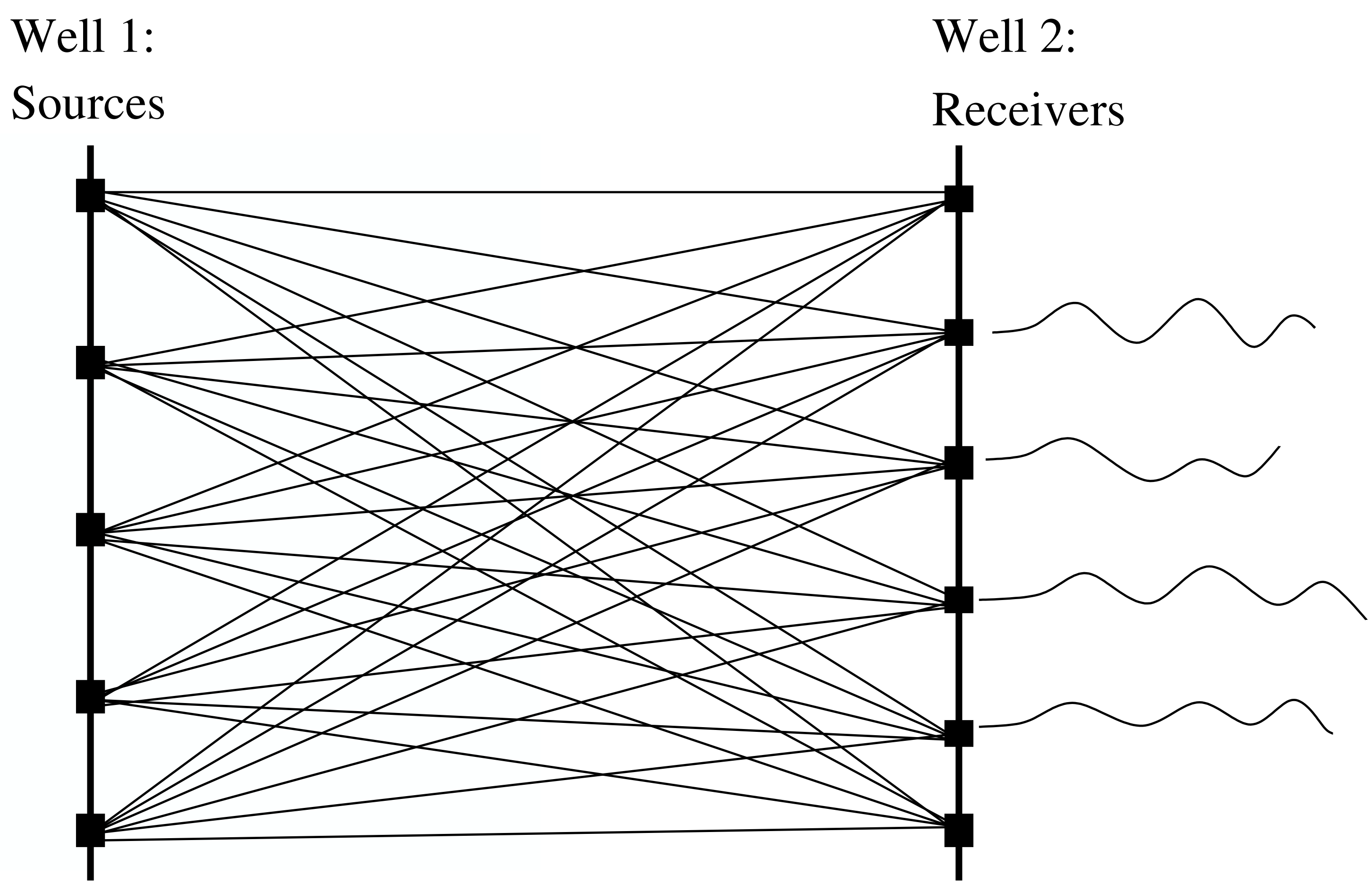
Outline

- Overview for traveltime tomography
- Part I: First-Arrival(FA) based traveltime tomography
  - Mismatching functional and adjoint state methods
  - Fast sweeping for eikonal eqns and adjoint eqns
  - Synthetic examples
- Part II: Multi-arrival(MA) based traveltime tomography
  - Paraxial Liouville equations for MAs
  - Mismatching functional and adjoint state methods
  - Examples
- Conclusions and future work

Transmission traveltime tomography



Ray-tracing based tomography

- Traveltime between S and R: $t(S,R) = \int_S^R \frac{ds}{c}$.
- Fermat's principle serves as the foundation: First-Arrivals (FA) based.
- Both ray path and velocity (1/slowness) are unknown.
- Linearize the equation around a given background slowness with an unknown slowness perturbation.
- Discretize the interested region into pixels of constant velocities.
- Trace rays in the Lagrangian framework.
- Obtain a linear system linking slowness perturbation with traveltime perturbation.

Seismic traveltime tomography

- Transmission traveltime tomography estimates wave-speed distribution from acoustic, elastic or electromagnetic first-arrival (FA) traveltime data.
- Travel-time tomography shares some similarities with medical X-ray CT.
- Geophysical traveltime tomography uses travel-time data between source and receiver to invert for underground wave velocity.
- Seismic tomography usually is formulated as a minimization problem that produces a velocity model minimizing the difference between traveltimes generated by tracing rays through the model and those measured from the data: Lagrangian approaches.

Traveltime tomography: PDE-based (1)

- We develop PDE-based Eulerian approaches to traveltime tomography to avoid ray-tracing.
- Part I: FA-based traveltime tomography via eikonal eqns, adjoint state methods and fast eikonal solvers.
  - Sei-Symes'94, '95 formulated FA based traveltime tomography using paraxial eikonal eqns; they only illustrated the feasibility of computing the gradient by using the adjoint state method.
  - Our contribution: formulating the problem in terms of the full eikonal eqn, solving the eikonal eqn by fast sweeping methods and designing a new fast sweeping method for the adjoint eqn of the linearized eikonal eqn.

Traveltime tomography: PDE-based (2)

- Part II: multi-arrival (MA)-based traveltime tomography via Liouville eqns and adjoint state methods.
  - Our contribution: to our knowledge this is the first Eulerian approach to taking into account all arrivals systematically in the seismic tomography.
  - Delprat-Jannaud and Lailly'95: handling multiple arrivals (MAs) in reflection tomography in the ray-tracing framework, a Lagrangian approach.

Part I: Eikonal-based tomography

- Traveltimes between a source S and receivers R on the boundary satisfy

$$c(\mathbf{x})|\nabla T| = 1, \;\; T(\mathbf{x_s}) = 0.$$

- Forward problem: given $c > 0$, compute the viscosity solution based FAs from the source to receivers.
- Inverse problem: given both FA measurements on the boundary $\partial\Omega_p$ and the location of the point source $\mathbf{x}_s \in \partial\Omega_p$, invert for the velocity field $c(\mathbf{x})$ inside the domain Ω_p.

FA-based tomography: idea

- Forward problem: fast eikonal solvers; they are essential for inverse problems.
- Inverse problem: essential steps.
  - Minimize the mismatching functional between measured and simulated traveltimes.
  - Derive the gradient of the mismatching functional and apply an optimization method.
  - Linearize the eikonal eqn around a known slowness with an unknown slowness perturbation.
  - Solve the eikonal eqn for the viscosity solution: only FAs are used.

FA-based tomography: formulation

- The mismatching functional (energy),

$$E(c) = \frac{1}{2} \int_{\partial\Omega_p} |T - T^*|^2 \,,$$

where $T^*|_{\partial\Omega_p}$ is the data and $T|_{\partial\Omega_p}$ is the eikonal solution.

- Perturb c by $\epsilon\tilde{c} \Rightarrow$ Perturbation in T by $\epsilon\tilde{T}$ and in E by δE:

$$\delta E = \epsilon \int_{\partial\Omega_p} \tilde{T}(T - T^*) + O(\epsilon^2) \,.$$

$$T_x\tilde{T}_x + T_y\tilde{T}_y + T_z\tilde{T}_z = -\frac{\tilde{c}}{c^3} \,.$$

- Difficulty: δE depends on $\tilde{c}$ implicitly through $\tilde{T}$ and the linearized eikonal equation. Use the adjoint state method.

FA-based tomography: adjoint state

- Introduce λ satisfying

$$[(-T_x)\lambda]_x + [(-T_y)\lambda]_y + [(-T_z)\lambda]_z = 0\,,$$
$$(\mathbf{n}\cdot\nabla T)\lambda = T^* - T, \text{on } \partial\Omega_p.$$

- Impose the BC to back-propagate the time residual into the computational domain.
- Simplify the energy perturbation further,

$$\frac{\delta E}{\epsilon} = \int_{\Omega_p} \frac{\tilde{c}\lambda}{c^3}\,.$$

- Choose $\tilde{c} = -\lambda/c^3 \Rightarrow$ Decrease the energy: $\delta E = -\epsilon \int_{\Omega_p} \tilde{c}^2 \leq 0$.

FA-based tomography: regularization

- Enforce
  1. $\tilde{c}|_{\partial\Omega_p} = 0$;
  2. $c^{k+1} = c^k + \epsilon\tilde{c}^k$ smooth.
- The first condition is reasonable as we know the velocity on the boundary.
- The second condition is a requirement on the smoothness of the update at each step.
- Regularize, $\nu \geq 0$, by using a Sobolev space,

$$\tilde{c} = -(I - \nu\Delta)^{-1}\left(\frac{\lambda}{c^3}\right),$$

$$\delta E = -\epsilon \int_{\Omega_p} (\tilde{c}^2 + \nu|\nabla\tilde{c}|^2) \leq 0\,.$$

FA-based tomography: multiple data sets (1)

- A single data set is associated with a single source.
- Incorporate multiple data sets associated with multiple sources into the formulation.
- Define a new energy for N sets of data:

$$E^N(c) = \frac{1}{2} \sum_{i=1}^{N} \int_{\partial \Omega_p} |T_i - T_i^*|^2,$$

where T_i are the solutions from the eikonal equation with the corresponding point source condition $T(\mathbf{x}_s^i) = 0$.

FA-based tomography: multiple data sets (2)

- Perturbation in the energy,

$$\frac{\delta E^N}{\epsilon} = \int_{\Omega_p} \frac{\tilde{c}}{c^3} \sum_{i=1}^{N} \lambda_i \, ,$$

where λ_i is the adjoint state of T_i ($i = 1, \cdots, N$) satisfying

$$\{[-(T_i)_x]\lambda_i\}_x + \{[-(T_i)_y]\lambda_i\}_y + \{[-(T_i)_z]\lambda_i\}_z = 0 \, ,$$
$$(\mathbf{n} \cdot \nabla T_i)\lambda_i = T_i^* - T_i \, .$$

- To minimize the energy $E^N(c)$, choose

$$\tilde{c} = -(I - \nu\Delta)^{-1} \left(\frac{1}{c^3} \sum^{N} \lambda_i \right) \, .$$

Fast sweeping for eikonal and adjoint equations

- Fast eikonal solvers: fast marching (Sethian, ...), ENO-DNO-Postsweeping (Kim-Cook), fast sweeping on Cartesian and triangular meshes (Zhao, Tsai, Cheng, Osher, Kao, Qian, Cecil, Zhang,...); see Engquist-Runborg'03 for more.
- The eikonal eqn is solved by the fast sweeping method (Zhao, Math. Comp'05).
- The adjoint equation for the adjoint state can be solved by fast sweeping methods as well.
- We have designed a new fast sweeping method for the adjoint eqn. (Leung-Qian'05)

Fast sweeping for the adjoint equation (1)

- Take the 2-D case to illustrate the idea:

$$(a\lambda)_x + (b\lambda)_z = 0\,,$$

  where a and b are given functions of (x, z).

- Consider a computational cell centered at (x_i, z_j) and discretize the equation in conservation form,

$$\begin{aligned} & \frac{1}{\Delta x}\left(a_{i+1/2,j}\lambda_{i+1/2,j} - a_{i-1/2,j}\lambda_{i-1/2,j}\right) \\ + \quad & \frac{1}{\Delta z}\left(b_{i,j+1/2}\lambda_{i,j+1/2} - b_{i,j-1/2}\lambda_{i,j-1/2}\right) = 0\,. \end{aligned}$$

Fast sweeping for the adjoint equation (2)

- λ on the interfaces, $\lambda_{i\pm 1/2,j}$ and $\lambda_{i,j\pm 1/2}$, determined by the propagation of characteristics, ie, upwinding,

$$
\begin{aligned}
& \frac{1}{\Delta x}\Big((a^{+}_{i+1/2,j}\lambda_{i,j} + a^{-}_{i+1/2,j}\lambda_{i+1,j})\Big) \\
- & \frac{1}{\Delta x}\Big((a^{+}_{i-1/2,j}\lambda_{i-1,j} + a^{-}_{i-1/2,j}\lambda_{i,j})\Big) \\
+ & \frac{1}{\Delta z}\Big((b^{+}_{i,j+1/2}\lambda_{i,j} - b^{-}_{i,j+1/2}\lambda_{i,j+1})\Big) \\
- & \frac{1}{\Delta z}\Big((b^{+}_{i,j+1/2}\lambda_{i,j-1} - b^{-}_{i,j+1/2}\lambda_{i,j})\Big) = 0\,,
\end{aligned}
$$

where $a^{\pm}_{i+1/2,j}$ denote the positive and negative parts of $a_{i+1/2,j}$.

Fast sweeping for the adjoint equation (3)

- Rewriting as

$$\alpha = \left(\frac{a^+_{i+1/2,j} - a^-_{i-1/2,j}}{\Delta x} + \frac{b^+_{i,j+1/2} - b^-_{i,j-1/2}}{\Delta z} \right)$$

$$\alpha \lambda_{i,j} = \frac{a^+_{i-1/2,j}\lambda_{i-1,j} - a^-_{i+1/2,j}\lambda_{i+1,j}}{\Delta x} + \frac{b^+_{i,j-1/2}\lambda_{i,j-1} - b^-_{i,j+1/2}\lambda_{i,j+1}}{\Delta z}$$

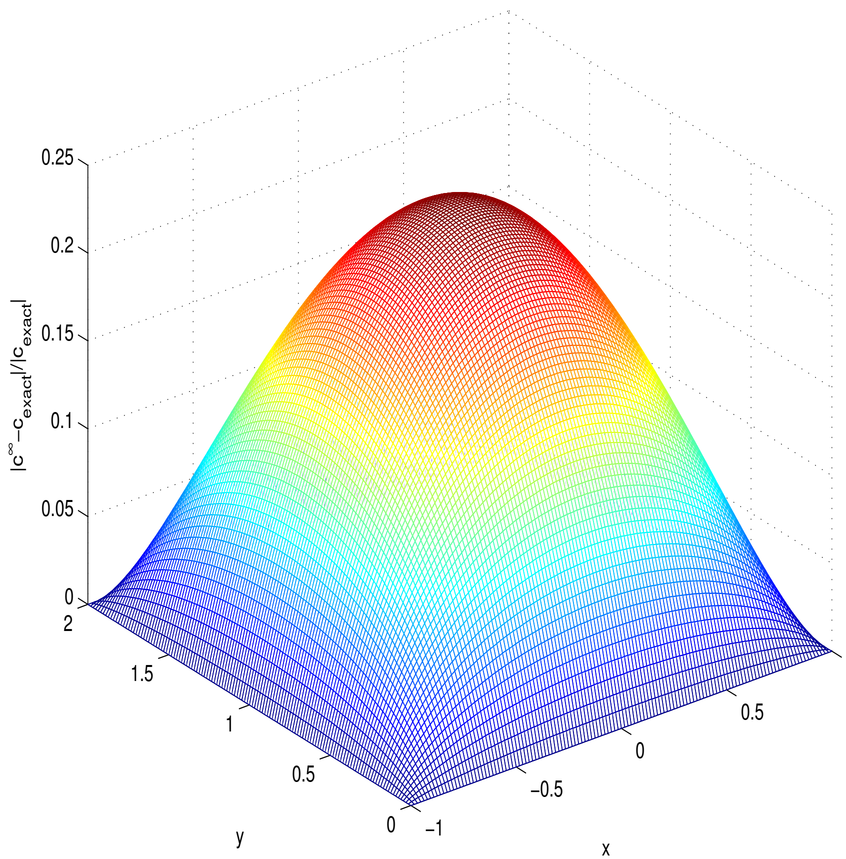
  which gives us an expression to construct a fast sweeping type method.

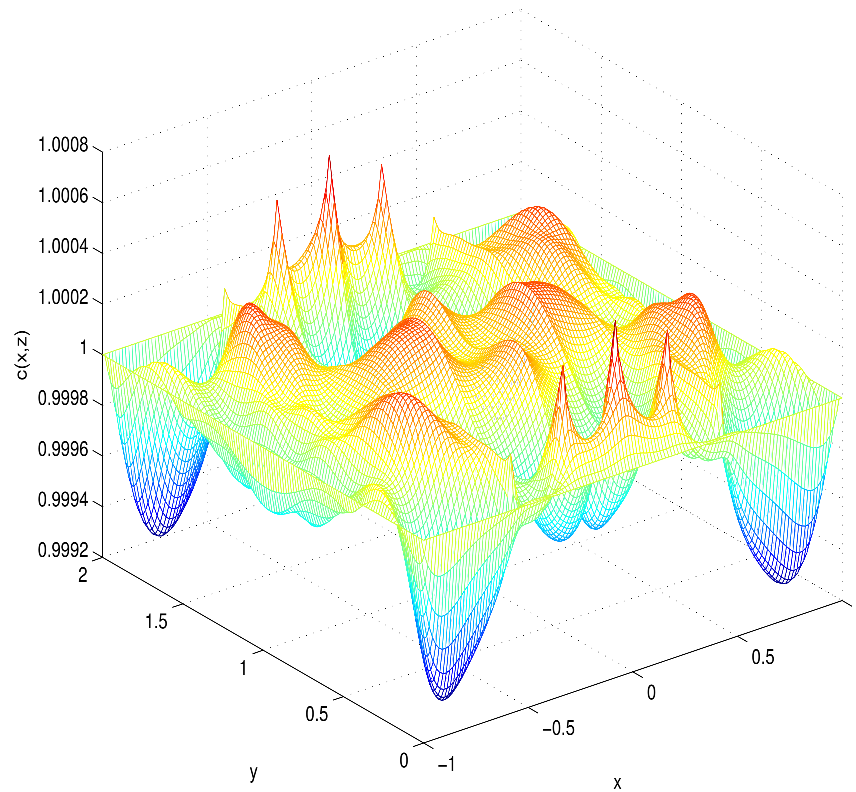
- Alternate sweeping strategy applies.

Other implementation details

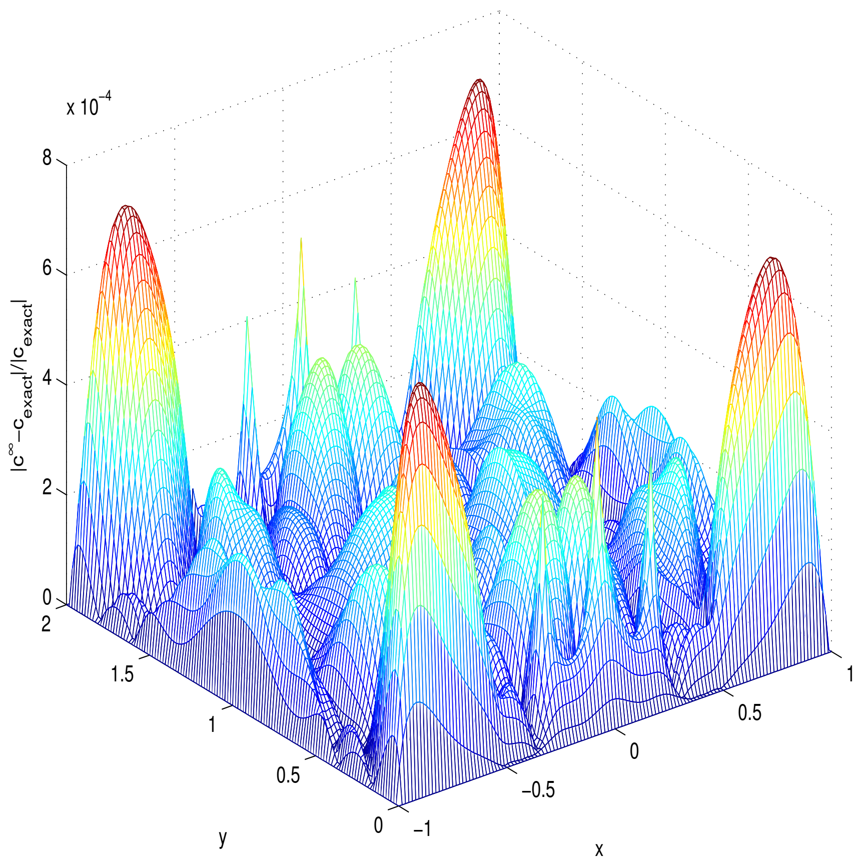
- The Poisson eqn is solved by FFT.
- The gradient descent method needs too many iterations.
- Use the limited memory Broyden, Fletcher, Goldfarb, Shanno (L-BFGS) method: a quasi-Newton optimization method (Byrd, Lu, Nocedal and Zhu'95).
- Ideal illuminations are assumed.

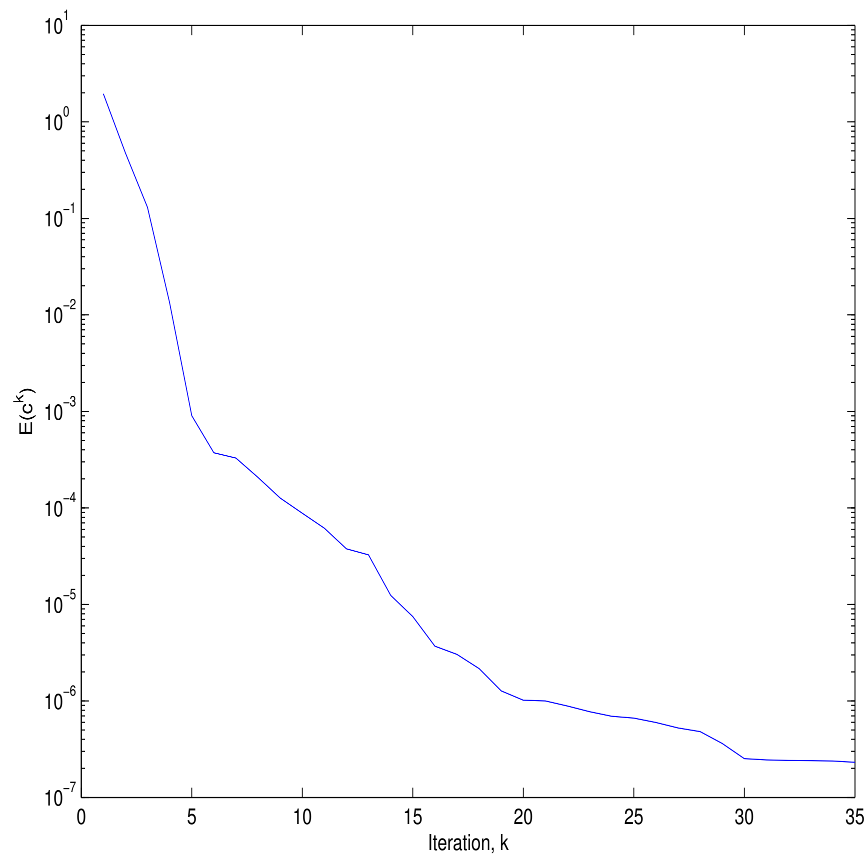
2-D Constant (1): 10 sources



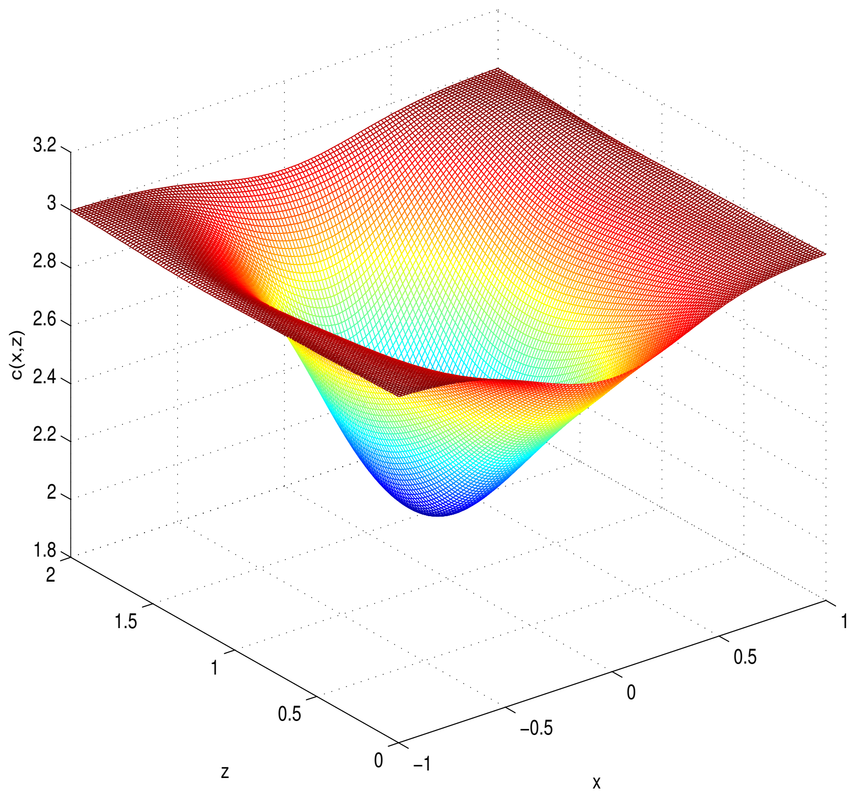


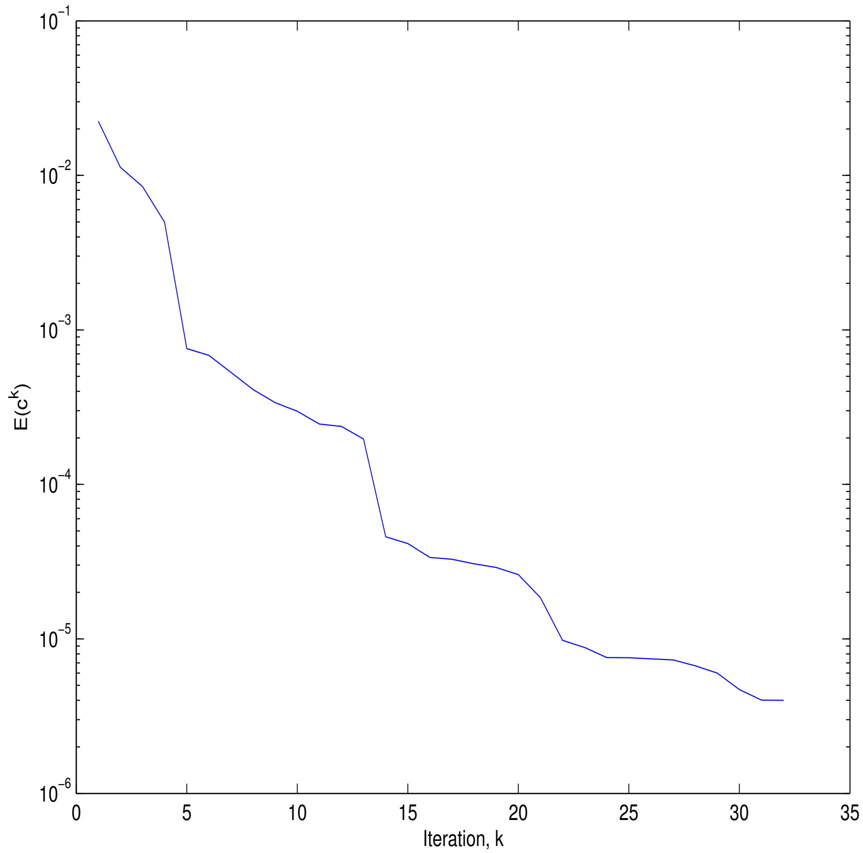
2-D Constant (2): 10 sources



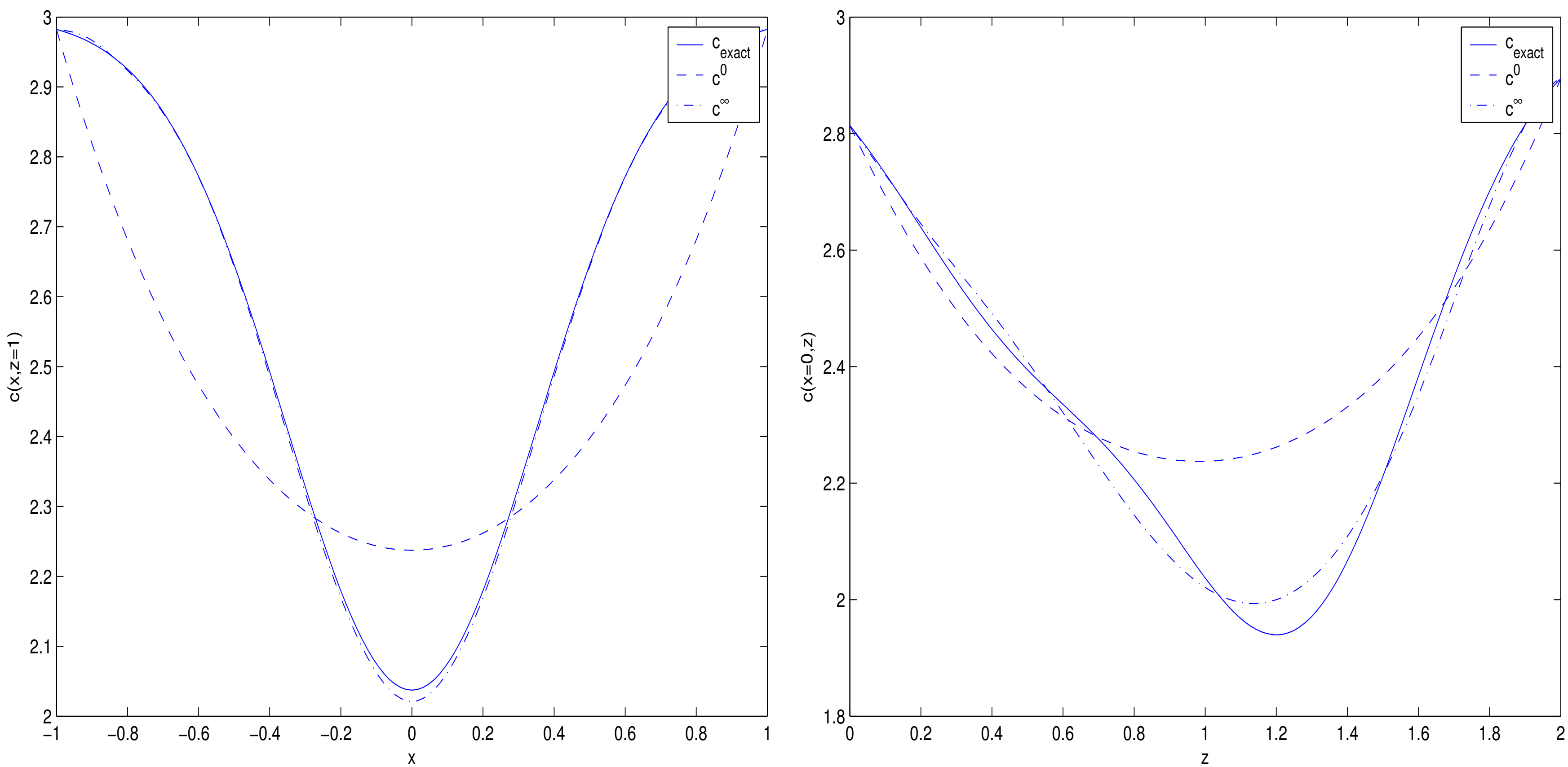


2-D two-Gaussian (1): 10 sources

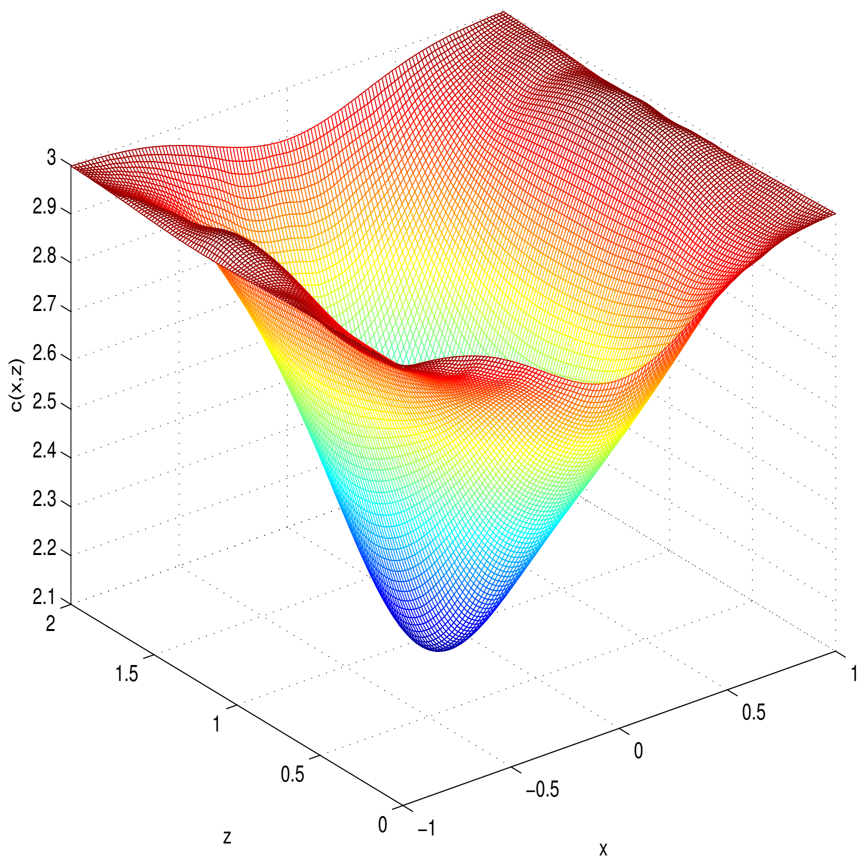


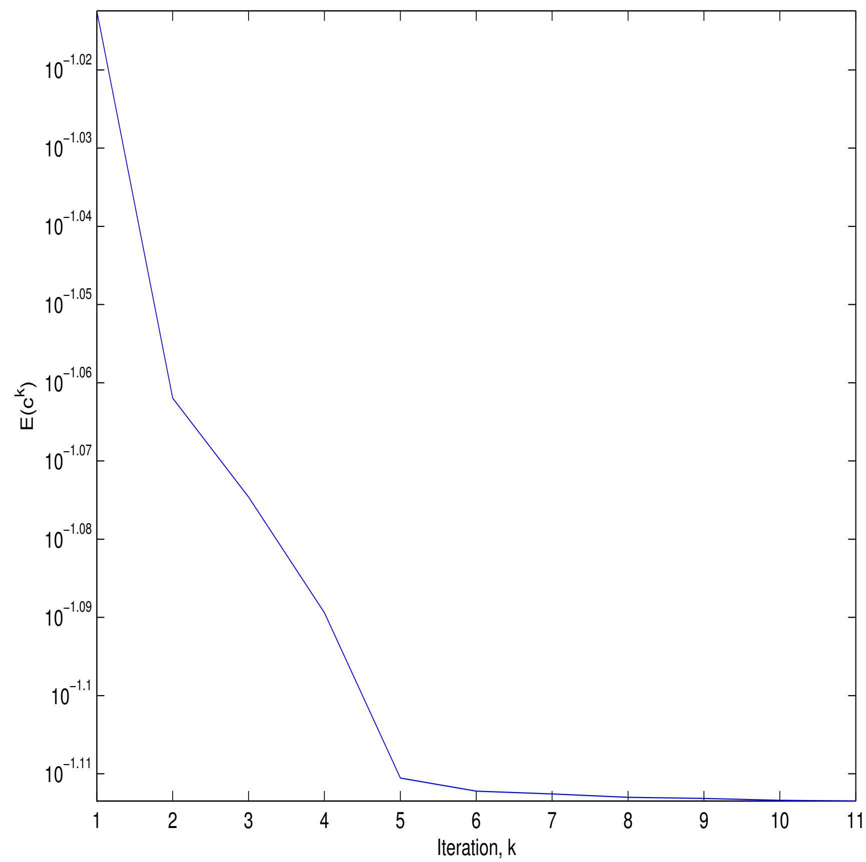


2-D two-Gaussian (2): 10 sources

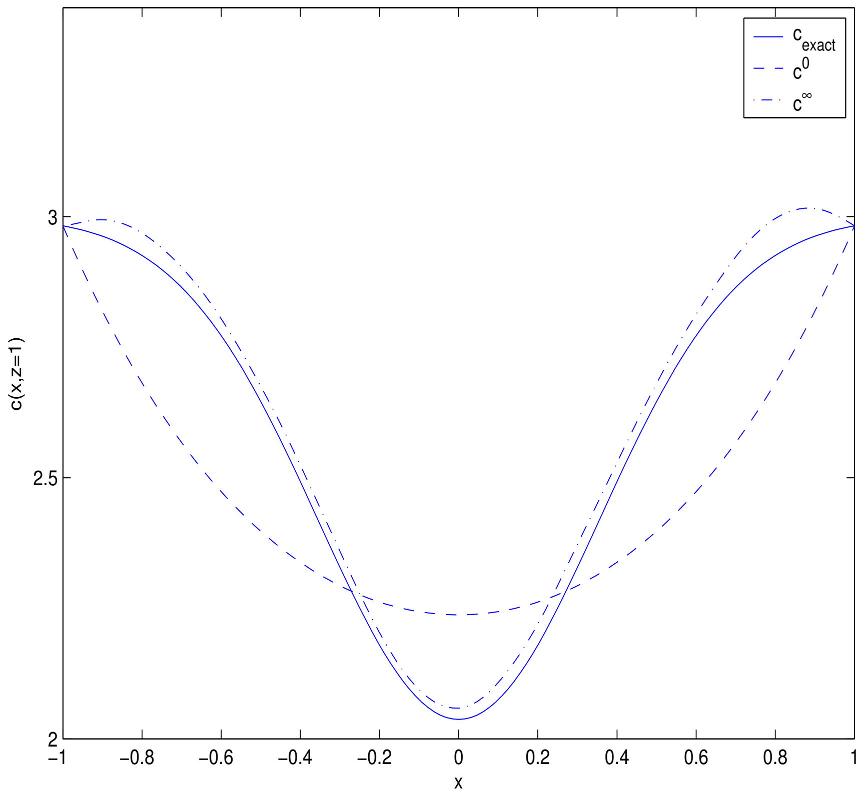


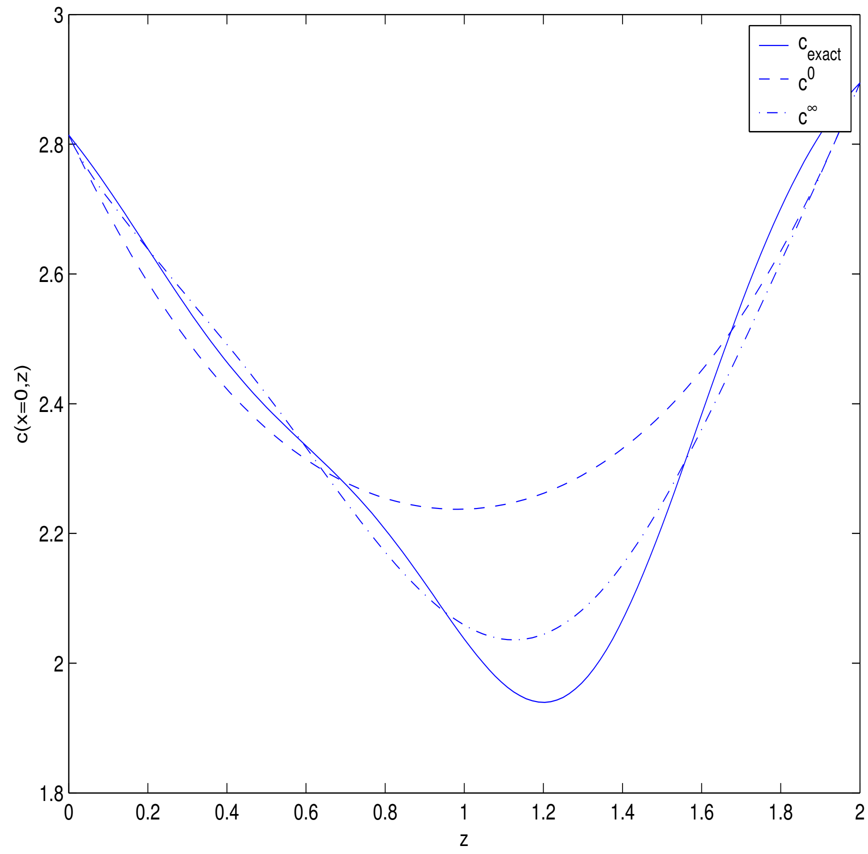
2-D two-Gaussian with 5% noise (1): 10 sources



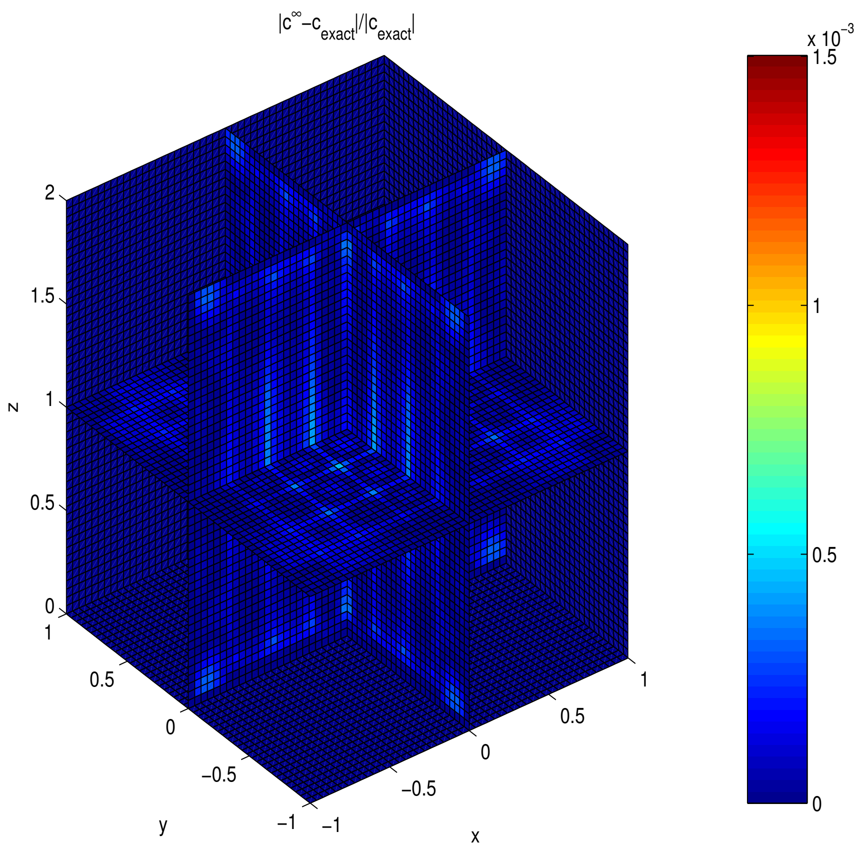


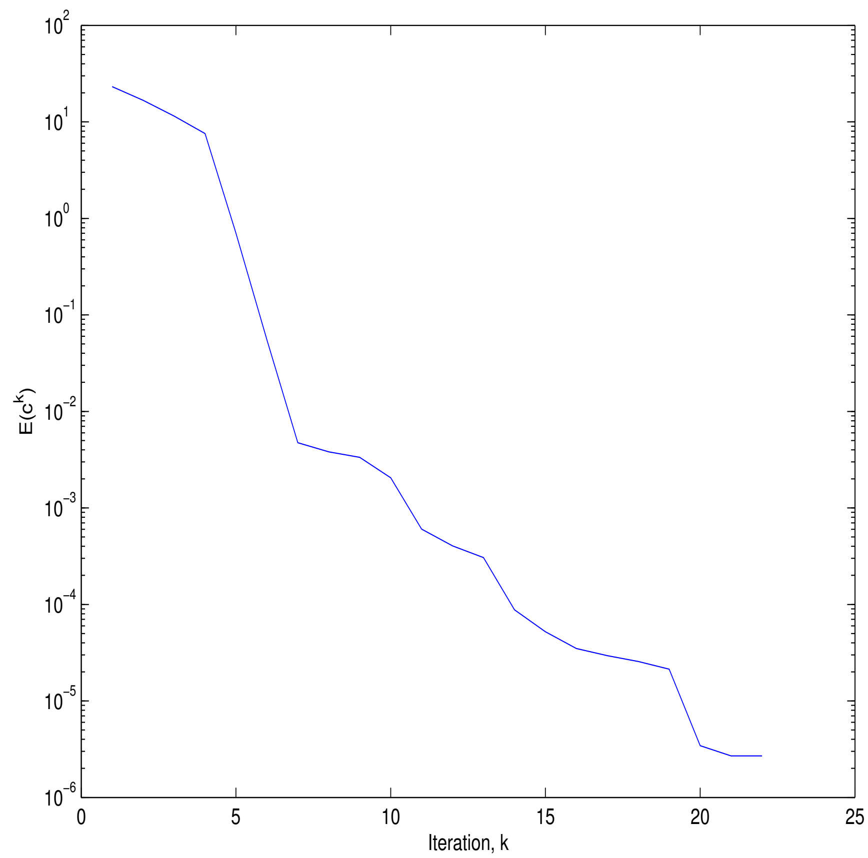
2-D two-Gaussian with 5% noise (2): 10 sources





3-D Constant: 98 sources





Marmousi: 20 sources

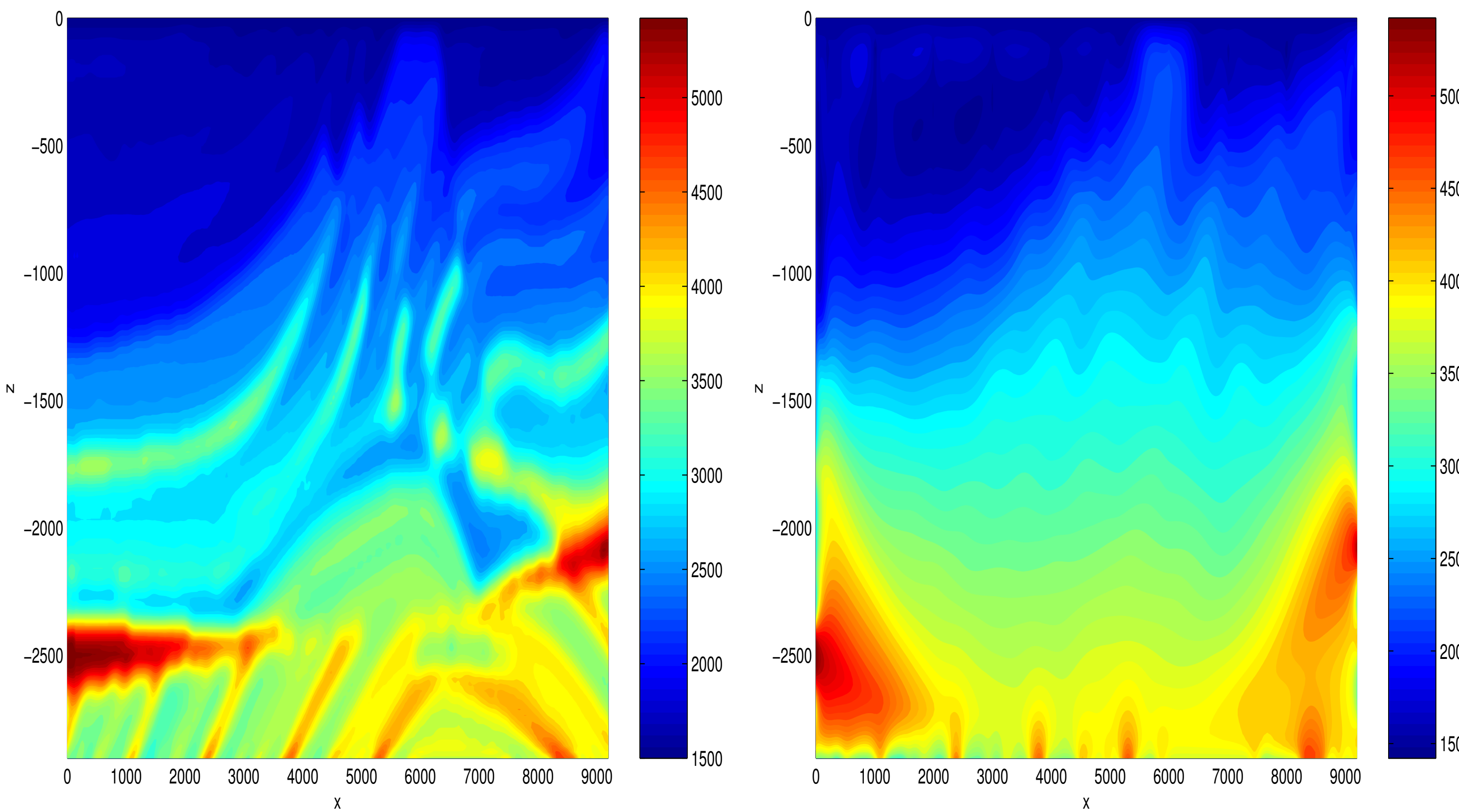


Figure 1: True synthetic Marmousi vs inversion

Marmousi: 20 sources

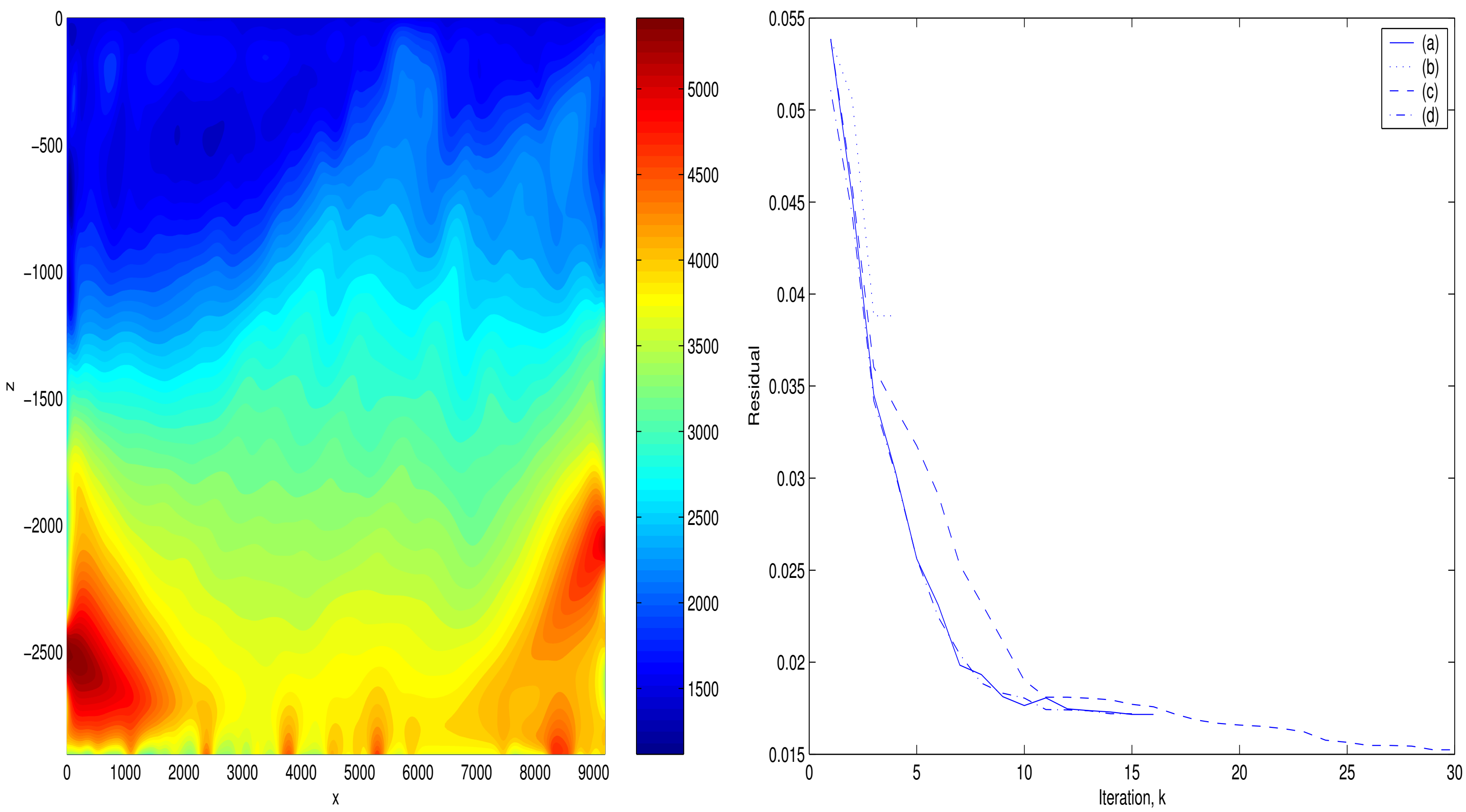


Figure 2: Refined mesh and residual history

Part II: Liouville-based tomography

- Question: can we take into account multi-arrivals (MA) to possibly improve resolution?
- Multi-arrival(MA) based traveltime tomography via Liouville eqns.
  - Liouville + Level set methods + adjoint state methods.
  - Our contribution: to our knowledge this is the first Eulerian approach to considering all arrivals systematically in the traveltime tomography.
  - Delprat-Jannaud and Lailly'95: handling multiple arrivals in reflection tomography in the ray-tracing framework: a Lagrangian approach.

Tomography via Liouville

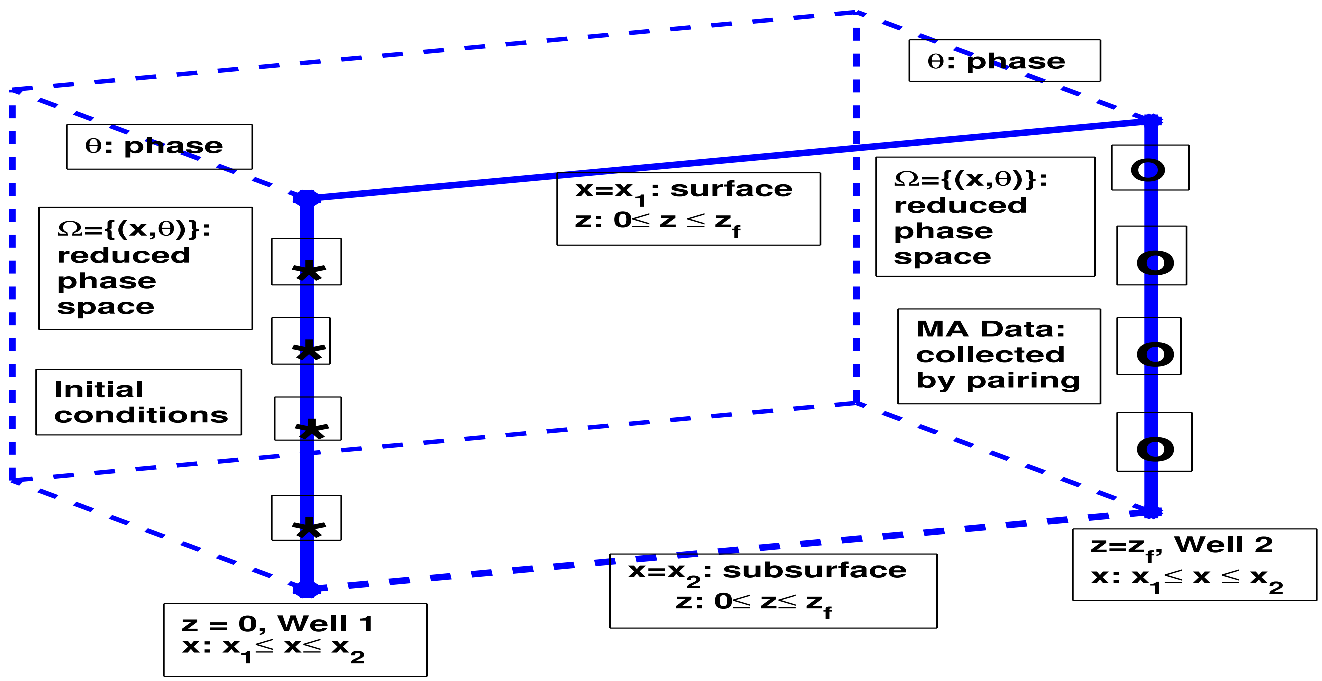


Figure 3: Use multi-arrivals from received time series via Liouville in phase space

MA tomography: Liouville

- Liouville based phase space geometrical optics (Engquist and Runborg'03; many others).
- Paraxial Liouville eqns are based on paraxial eikonals and level sets (Leung-Qian-Osher'04):

$$\frac{\partial \tau}{\partial z} = \sqrt{\max\left(\frac{1}{c^2} - \left(\frac{\partial \tau}{\partial x}\right)^2, \frac{\cos^2\theta_{\max}}{c^2}\right)},$$

$$\phi_z + u\phi_x + v\phi_\theta = 0,$$

$$T_z + uT_x + vT_\theta = \frac{1}{c\cos\theta},$$

where $\mathbf{u} = (u, v) = (\tan\theta, m_z \tan\theta - m_x)$, $m = m(c) = \log c$; ϕ and T are the level set and traveltime functions in the reduced phase space $\Omega = \{(x,\theta) : x_{\min} \le x \le x_{\max}, -\theta_{\max} \le \theta \le \theta_{\max}\}$.

MA tomography: complete data

- I.B.C. ($\mathbf{n}$ being the outward normal of $\partial\Omega$):

$$
\begin{aligned}
\phi(z_0,\cdot,\cdot) &= x \\
\phi(z,\cdot,\cdot)|_{\partial\Omega} &= \begin{cases} \phi^* & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) < 0 \\ \text{no b.c. needed} & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) \geq 0 \end{cases} \\
T(z_0,\cdot,\cdot) &= 0 \\
T(z,\cdot,\cdot)|_{\partial\Omega} &= \begin{cases} T^* & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) < 0 \\ \text{no b.c. needed} & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) \geq 0 \end{cases} .
\end{aligned}
$$

- Use $(\cdot)^*$ to denote the measured value on the outflow boundary of $\partial\Omega$ and on the final level $z = z_f$.
- Such measurements can be picked by suitably pairing as in Delprat-Jannaud and Lailly'95.

MA tomography: energy

- $\tilde{\Omega} = \Omega \times (z_0, z_f)$; $\Omega_p = (x_{\min}, x_{\max}) \times (z_0, z_f)$.
- Data: $\phi^*(z,\cdot,\cdot)|_{\partial\Omega}$ and $T^*(z,\cdot,\cdot)|_{\partial\Omega}$ on the outflow boundary; $\phi^*(z_f,\cdot,\cdot)$ and $T^*(z_f,\cdot,\cdot)$ at $z = z_f$; $m|_{\partial\Omega_p}$.
- Minimize the energy:

$$E(m) = \frac{1}{2}\int_\Omega (\phi - \phi^*)^2|_{z=z_f} + \frac{1}{2}\int_z \int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})(\phi - \phi^*)^2$$
$$+\frac{\beta}{2}\int_\Omega (T - T^*)^2|_{z=z_f} + \frac{\beta}{2}\int_z \int_{\partial\Omega} (\mathbf{u}\cdot\mathbf{n})(T - T^*)^2.$$

- Derive the gradient of the nonlinear functional by the adjoint state method.
- Linearize the Liouville eqns and the energy around a known background slowness with an unknown slowness perturbation.

MA tomography: linearization

- Perturb m by $\epsilon\tilde{m}$; changes in ϕ and T by $\epsilon\tilde{\phi}$ and $\epsilon\tilde{T}$:

$$\tilde{\phi}_z + u\tilde{\phi}_x + v\tilde{\phi}_\theta = [\tilde{m}_x - \tilde{m}_z \tan\theta]\phi_\theta,$$
$$\tilde{T}_z + u\tilde{T}_x + v\tilde{T}_\theta = [\tilde{m}_x - \tilde{m}_z \tan\theta]T_\theta - \frac{\tilde{m}}{c\cos\theta}\,.$$

- Perturbation in energy:

$$\delta E = E(m + \epsilon\tilde{m}) - E(m),$$

where $\tilde{\mathbf{u}} = (0, \tilde{v}) = (0, \tilde{m}_z \tan\theta - \tilde{m}_x)$.

MA tomography: adjoints

- Choose λ_1 and λ_2 such that

$$(\lambda_1)_z + (u\lambda_1)_x + (v\lambda_1)_\theta = 0,$$
$$(\lambda_2)_z + (u\lambda_2)_x + (v\lambda_2)_\theta = 0\,,$$

with the “initial” conditions on $z = z_f$,

$$\lambda_1(z = z_f) = \phi^* - \phi$$
$$\lambda_2(z = z_f) = T^* - T\,.$$

- With boundary conditions ...

MA tomography: gradient

- Boundary conditions

$$\lambda_1|_{\partial\Omega} = \begin{cases} \phi^* - \phi & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) > 0 \\ \text{no b.c. needed} & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) \leq 0 \end{cases}$$

$$\lambda_2|_{\partial\Omega} = \begin{cases} T^* - T & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) > 0 \\ \text{no b.c. needed} & \text{if} \quad (\mathbf{u}\cdot\mathbf{n}) \leq 0 \end{cases}$$

- Perturbation in energy (f_i, i=1:4 computable):

$$\delta E = \epsilon \int_{\Omega_p} \tilde{m} \left\{ (f_1)_x - (f_2)_z + \frac{\beta}{c} f_3 + f_4 \right\} .$$

MA tomography: regularization

- To decrease the energy, choose by Tikhonov regularization

$$\tilde{m} = -(I - \nu\Delta)^{-1} g$$

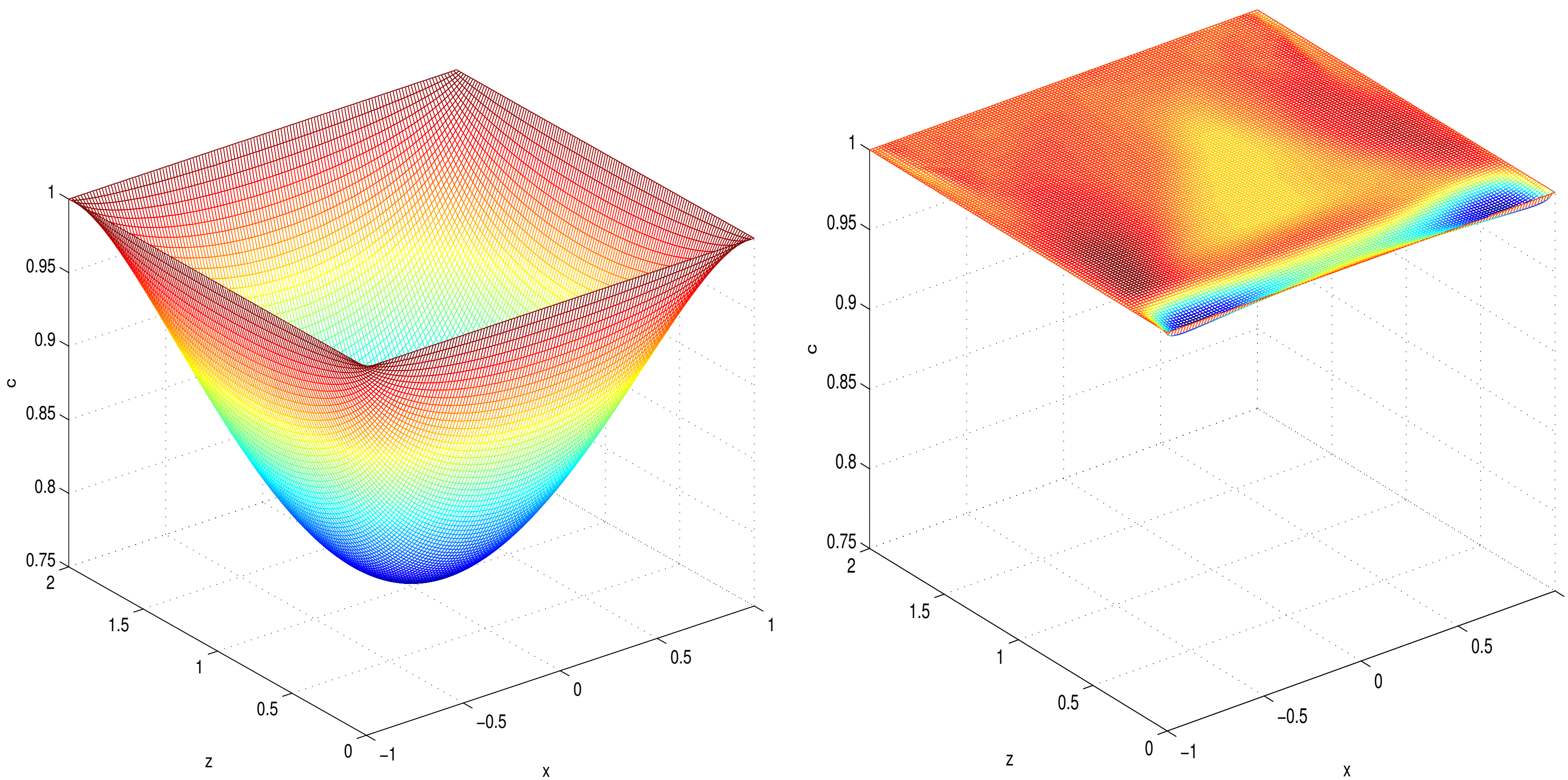
where

$$g = (f_1)_x - (f_2)_z + \frac{\beta}{c} f_3 + f_4.$$

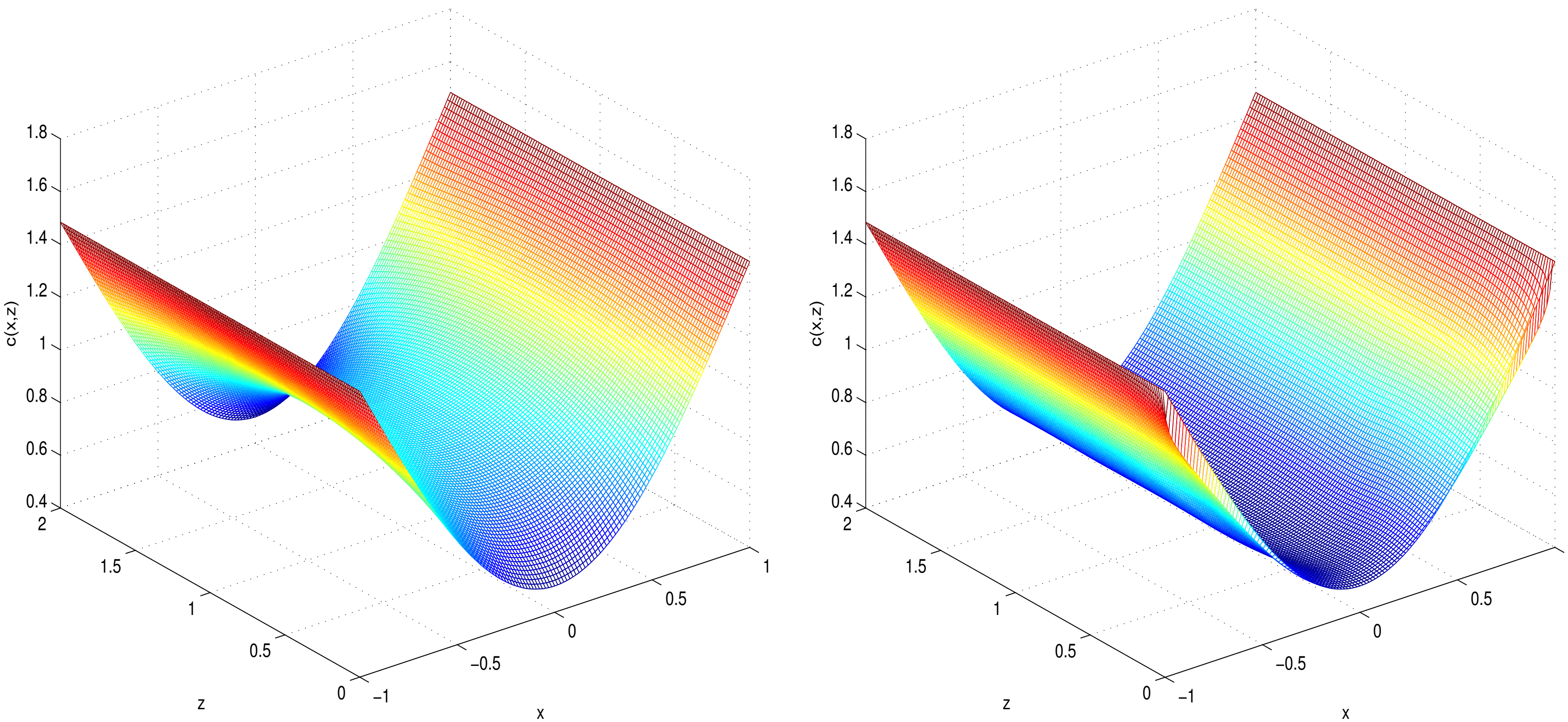
- $$\delta E = \epsilon \int_{\Omega_p} \tilde{m} g = -\epsilon \int_{\Omega_p} (|\tilde{m}|^2 + \nu |\nabla \tilde{m}|^2) \leq 0.$$

- Implementations: HJ-WENO, HJ-Central-WENO, FFT and gradient descent methods.

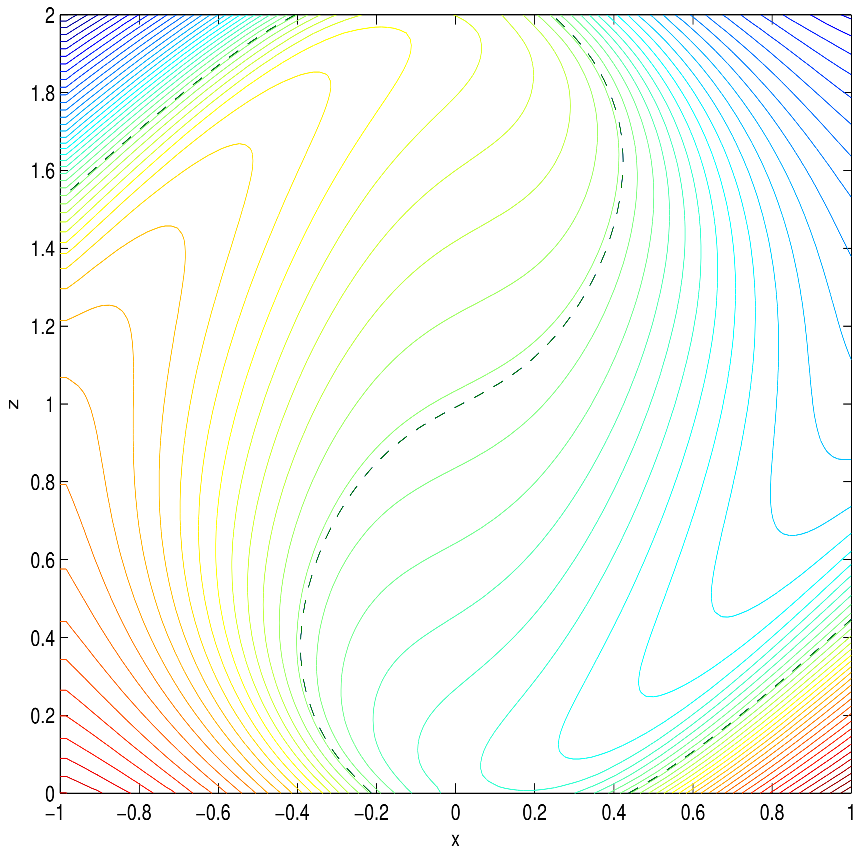
Constant vel.

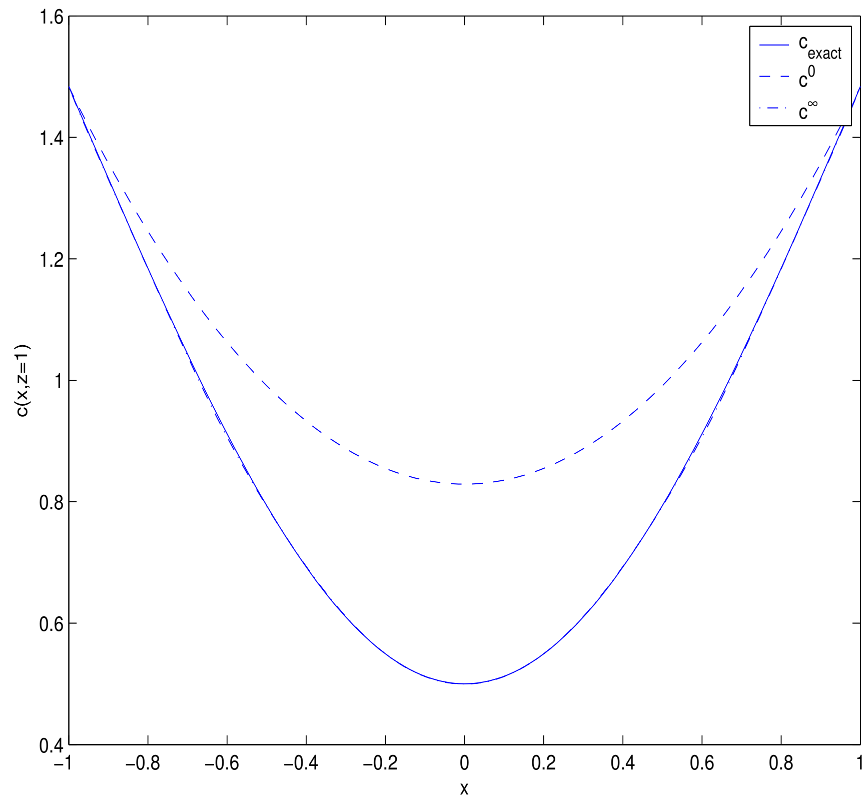


Waveguide vel.: (1)

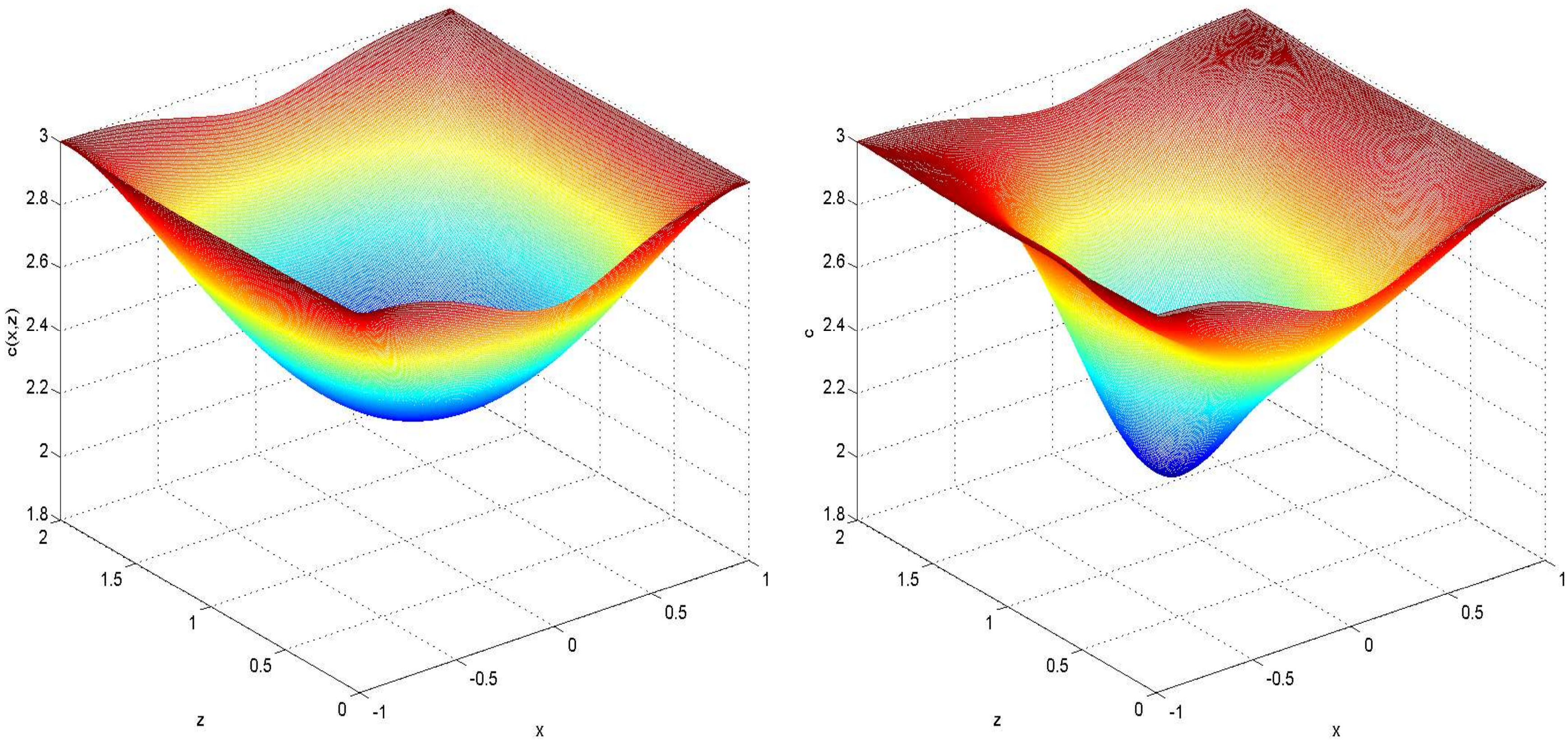


Waveguide vel.: (2)

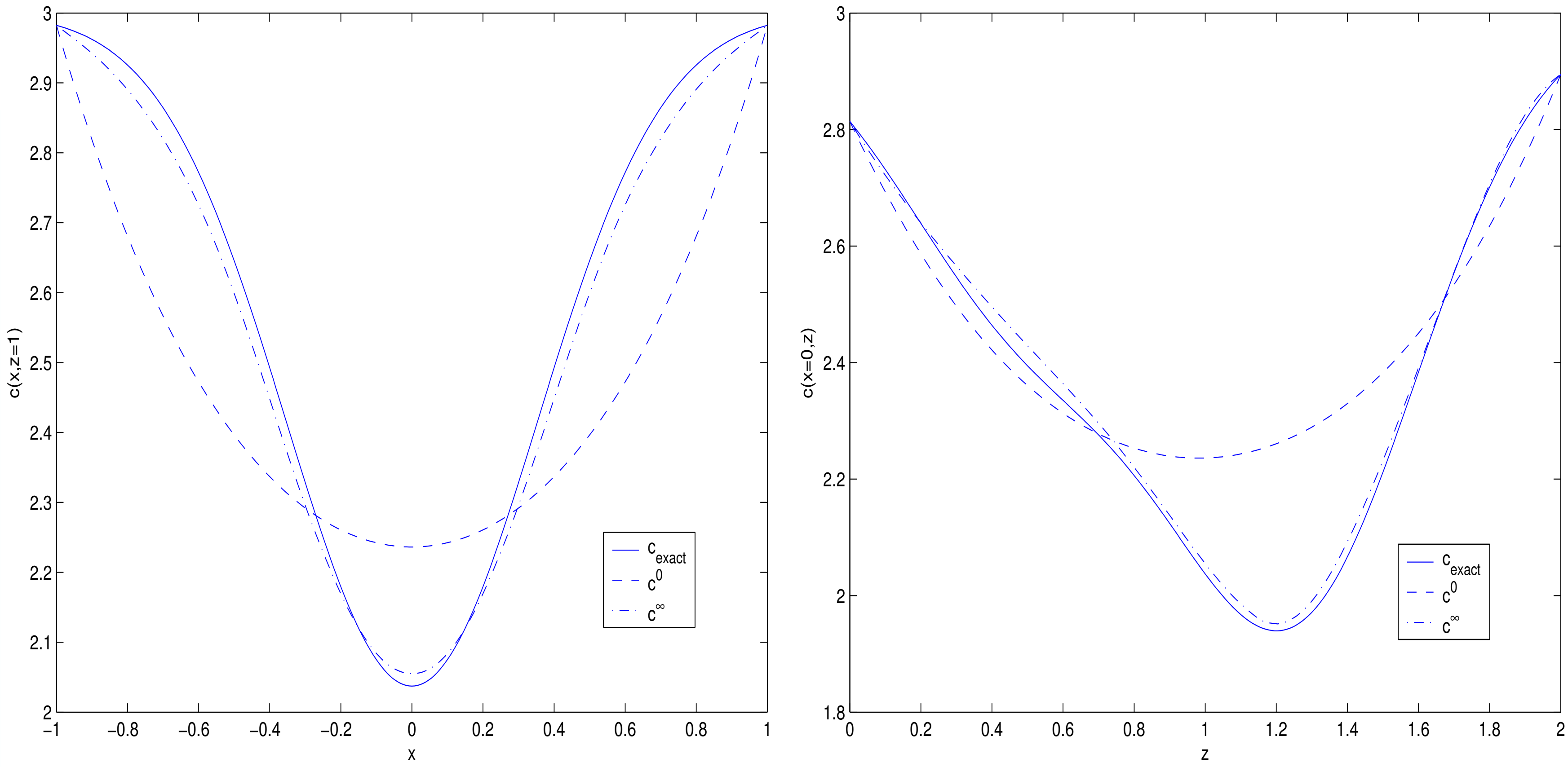




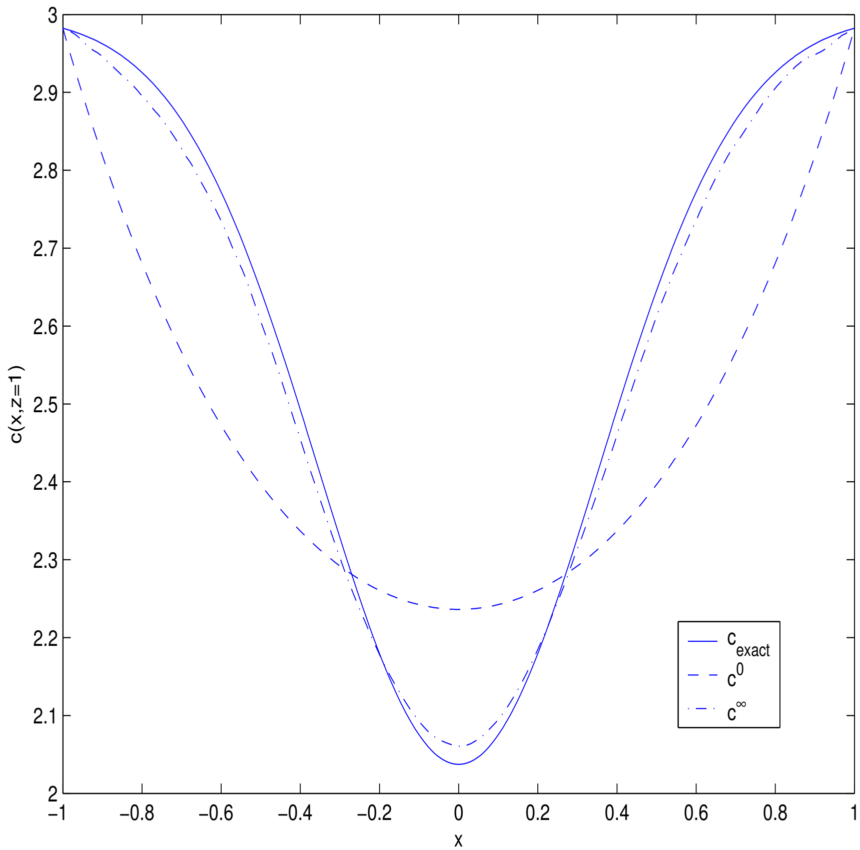
2-D two-Gaussian vel.: (1)

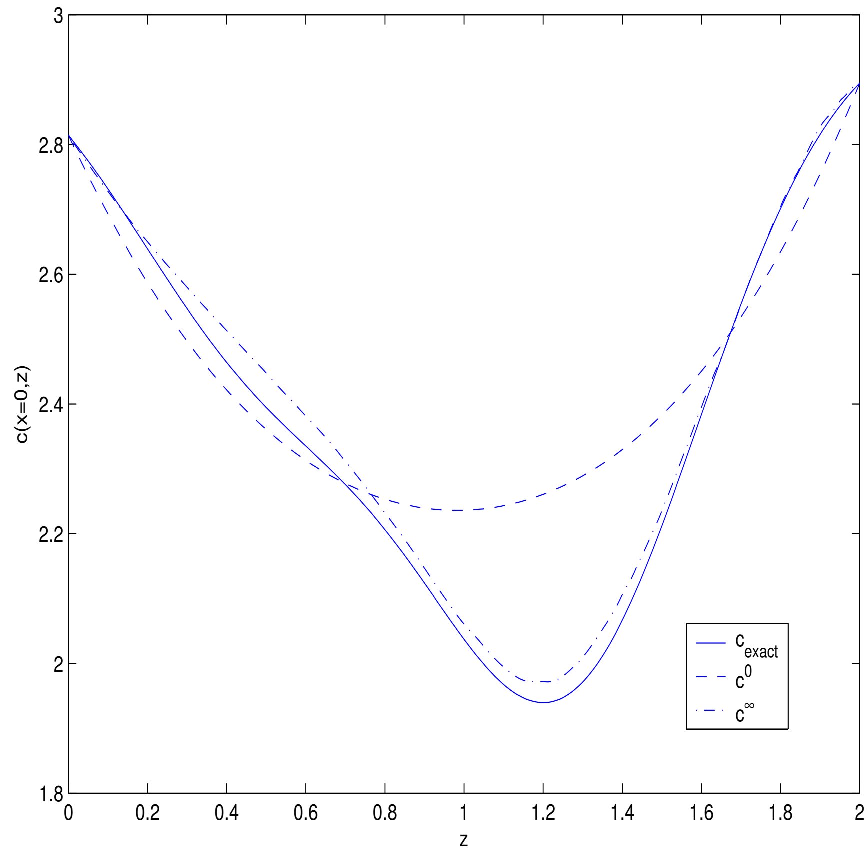


2-D two-Gaussian vel.: (2)



Two-Gaussian vel. with noisy data





Two-Gaussian: FA vs MA

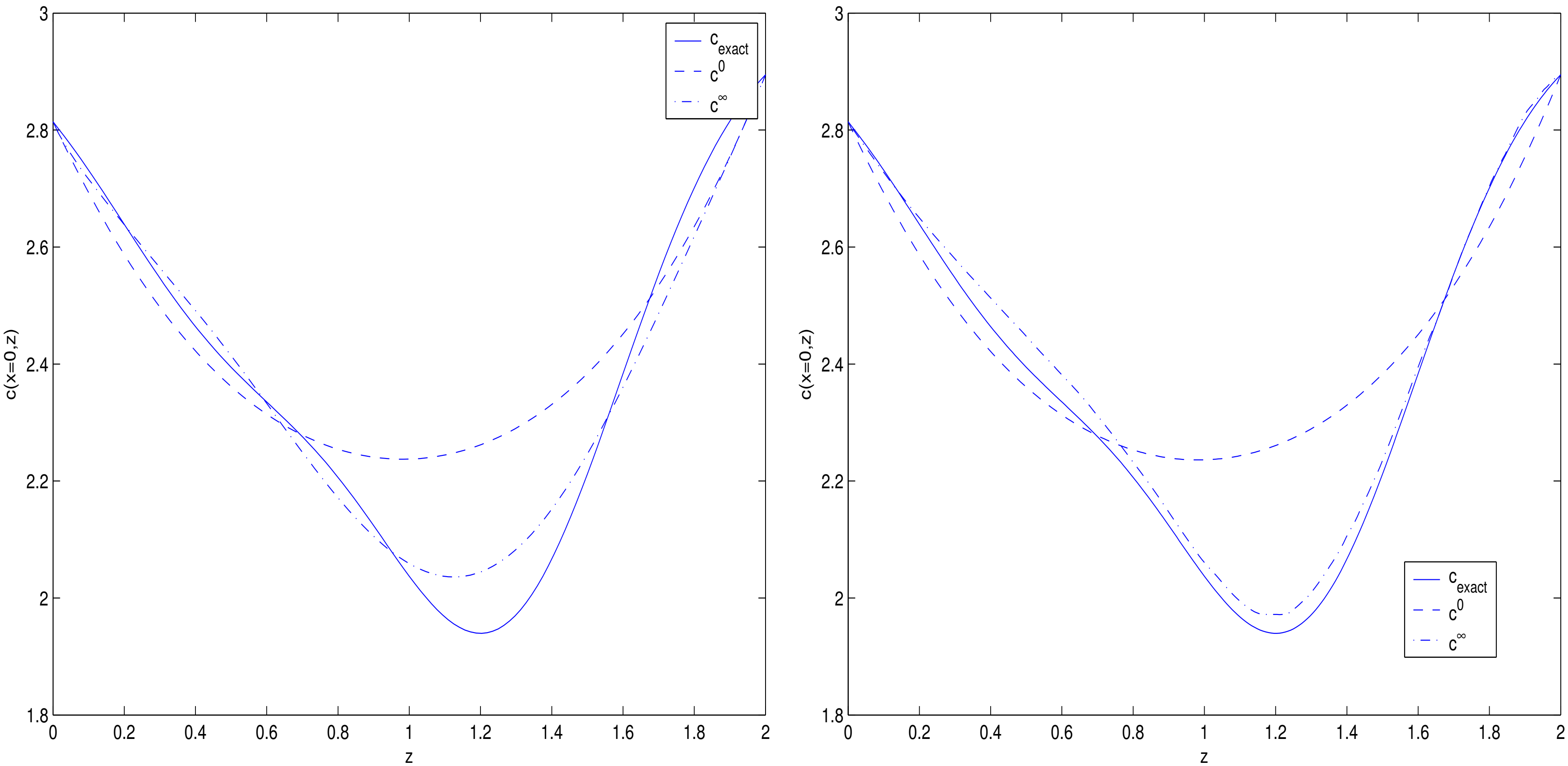


Figure 4: (a): FA with 10 sources; (b): MA.

MA tomography: incomplete data

- Only have the measurement on the final level $z = z_f$.
- Data: $\Gamma(z_f) = \{\phi(x,\theta,z_f), T(x,\theta,z_f) : (x,\theta) \in \Omega\}$.
- Paraxial assumption implies that relevant rays will not touch the boundary of the domain $\tilde{\Omega} = \Omega \times (0, z_f)$.
- Ignore the contribution from inflows in the energy.
- Simplify the energy:

$$
\begin{aligned}
E(m) \;&=\; \frac{1}{2}\int_{\Omega}(\phi-\phi^*)^2\delta(\Gamma(z_f)) + \\
&\quad + \frac{\beta}{2}\int_{\Omega}(T-T^*)^2\delta(\Gamma(z_f))\,.
\end{aligned}
$$

- Simplify the gradient as well.

Conclusion and future work

- Developed PDE-based approaches to traveltime tomography: FAs and MAs.
- Validated accuracy and efficiency of the approaches under ideal illuminations.
- Future work consists of
  - taking into account the partial illumination of the computational domain (Joint with TRIP);
  - formulating FA-based reflective traveltime tomography (Leung-Qian’05)
  - formulating MA-based high resolution reflective traveltime tomography
  - ...