Theory of differential semblance velocity analysis by wave-equation migration

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Summary

Prestack wave equation migration using the double square root equation produces prestack image volumes free of artifacts, even in the presence of multipathing due to complex structure. In particular image gathers in angle or offset ray parameter are flat at correct velocity, and gathers in offset are concentrated at zero offset. Differential semblance measures the deviation from flatness or concentration, and provides a method of automatic velocity updating via optimization. The adjoint state method gives a convenient computation of the differential semblance gradient as an addendum to prestack depth extrapolation.

Introduction

Migration methods based on one way depth extrapolation or downward continuation of data have a long history in seismic imaging, see e.g. (Claerbout, 1985) and references cited there. When data is given at all midpoints and offsets, then an approximation of data that would have been measured at some greater depth can be obtained by solving the so called double-square-root ("DSR") equation, a one-way wave equation in both source and receiver coordinates (this is the so-called "survey sinking" concept). In general of course not all data is available, but in this case data is still given by upward continuation, followed by restriction to acquisition set. Depth imaging using downward continuation is then essentially the adjoint of the upward continuation. It has been observed that various relaxations of the usual imaging condition permit the construction of prestack image volumes, in particular so-called image gathers, by one-way wave equation methods (Claerbout, 1985; de Bruin et al., 1990; Prucha et al., 1999; Sava et al., 2001).

Depth migration requires a velocity model, and prestack depth image volumes have characteristics which assist in velocity model construction. The *semblance principle* states that prestack image gathers are flat (or, in some cases, well-focused) when velocity models are correctly chosen. This principle underlies all practical velocity analysis methods based on prestack data, from NMO velocity spectra to migrated-domain tomography. The principle is not valid in complete generality, however. Strongly refracting models ("complex structure") generate multiple ray paths from sources and/or receivers to scattering points, and these in turn lead to *imaging artifacts*, i.e. coherent energy not corresponding to actual reflectors, for many common imaging methods. As a result, the image gathers created by common migration techniques are not flat even when the velocity is kinematically correct, and the semblance principle is violated. This phenomenon was first demonstrated and explained by Nolan (Nolan and Symes, 1996; Nolan and Symes, 1997), using two-way reverse time common shot imaging; artifacts also occur generically in image gathers produced by Kirchhoff common shot, common offset, and common scattering angle migration (Stolk, 2001; Stolk and Symes, 2002).

Wave equation migration can equally well serve as the basis for velocity analysis (Claerbout, 1985). The present paper is motivated by the recent discovery (Stolk and De Hoop, 2001) that wave equation imaging based on the DSR equation yields artifact-free images even in the presence of severe multipathing, and thus that the semblance principle holds for DSR migration. This fact guarantees that certain functions constructed from image gathers vary smoothly with velocity and attain their global minimum at kinematically correct models. These *differential semblance* ("DS") functions thus provide a systematic method for updating velocity models via optimization (Symes, 1986; Symes and Carazzone, 1991; Symes, 1998; Chauris and Noble, 2001; Mulder and ten Kroode, 2002; Brandsberg-Dahl et al., 2002). The version of DS introduced here has these properties for general velocity models, constrained only by the assumptions underlying migration in general (single scattering, velocity slowly varying on the wavelength scale) and DSR migration in particular (rays involved in imaging are nowhere horizontal). The DS-DSR combination should provide constructive velocity updates even in structurally complex, highly refracting zones such as salt flanks and chalk bodies, precisely the situations in which optimization based velocity analysis is potentially the most useful.

This paper gives precise definitions for several versions of differential semblance based on DSR migration, and describes an economical method for computing the DS gradient as an extension of the depth extrapolation process. The gradient, computed as described here, provides a search direction in velocity space for a modest additional computational cost beyond that for a migration.

Wave equation common image gathers

Our computations are valid both for 2 or 3 space dimensions. Horizontal (or lateral) position will be denoted by x (a 1- or 2-vector, respectively), or when it refers to a source or receiver position by s or r. Depth will be denoted by z. The medium velocity is $c = c(x, z)$. The frequency variable for time t is ω.

Wave-equation migration is based on downward continuation of data $d(s, r, t)$ or $d(s, r, \omega)$. The downward continued field $u(s, r, \omega, z)$ satisfies the DSR wave equation

(Claerbout, 1985):

$$\frac{\partial u}{\partial z} = F[c]u,\ u|_{z=0} = d \tag{1}$$

where $F[c]$ is the DSR operator

$$F[c] = -\mathrm{i}\omega\sqrt{c(s,z)^{-2} - \omega^{-2}D_s^2} - \mathrm{i}\omega\sqrt{c(r,z)^{-2} - \omega^{-2}D_r^2}. \tag{2}$$

in which $D_s = -\mathrm{i}\frac{\partial}{\partial s}$ (multiplication by k_s in Fourier domain), etc.

If the acquisition geometry is incomplete (sparsely sampled source-receiver pairs), as is usually the case, $d(s,r,t)$ is understood to be padded by zero traces to complete the sampling in s,r. Of course then the downward continued field cannot be the *physical* reflected field, as the necessary surface data is lacking: that is, it is not possible to literally "sink the survey". Instead, as explained in (Stolk and De Hoop, 2001), the DSR migration operator is the *adjoint* of the DSR (prestack) forward modeling operator, whether the acquisition geometry is complete or not. For this reason, the migration output has its high frequency energy in the correct places, hence produces a structural image, even though the downward continued field is not literally the data that would have been measured with sources and receivers sunk into the Earth.

Note that the critical assumption in the derivation of this equation is that rays carrying incident energy from source to reflection point, and reflected energy from reflection point to receiver, do not have horizontal tangents, i.e. along such rays $|\partial z/\partial t| \neq 0$. In fact to assure stability of depth extrapolation waves traveling near horizontally must be suppressed. This suppression is built into numerical realizations of DSR migration.

The depth image at (x,z) is extracted from the downward continued field by setting $s = r = x$ and $t = 0$ (Claerbout, 1985). This imaging condition does not address the formation of prestack volumes, however. Several definitions have been suggested for common image gathers, differing in the choice of gather parameter and method of formation:

0. (Claerbout, 1985): parametrized by half offset h, via restriction to $t = 0$:

$$V_0(x,z,h) = u(x+h, x-h, 0, z)$$

1. (de Bruin et al., 1990; Prucha et al., 1999): parametrized by offset ray parameter p, via Radon transform in the offset/time, followed by restriction to $t = 0$:

$$V_1(x,z,p) = (2\pi)^{-1}\times \int\int u(x+h, x-h, \omega, z)\mathrm{e}^{-i\omega p\cdot h}\phi(h)\,\mathrm{d}h\,\mathrm{d}\omega. \tag{3}$$

2. (Sava et al., 2001): parametrized by offset/depth slope q, via restriction to $t = 0$ followed by Radon transform in depth and offset:

$$V_2(x,z,q) = (2\pi)^{-1}\times \int\int u(x+h, x-h, \omega, z+qh)\phi(h)\,\mathrm{d}h\,\mathrm{d}\omega.$$

Here $\phi(h)$ is a cutoff function (mute). [**NB:** this section (as well as V_1) can be viewed as function of reflection angle rather than slope (or offset ray parameter), via simple relations between these quantities; see (Sava et al., 2001).]

Wave-equation differential semblance

Each of the prestack gather definitions enumerated in the last section has its characteristic signature of kinematically correct velocity:

0. $V_0(x,z,h)$ is focused, or concentrated, at $h = 0$, so $hV_0(x,z,h) \simeq 0$;

1. $V_1(x,z,p)$ is essentially p-independent, i.e. the fixed-x ("image") gathers are flat in p, so

$$\frac{\partial V_1}{\partial p}(x,z,p) \simeq 0$$

2. $V_2(x,z,q)$ is essentially q-independent, i.e. the fixed-x gathers are flat in q, so

$$\frac{\partial V_2}{\partial q}(x,z,q) \simeq 0$$

These criteria lead to three closely related differential semblance functions, which vanish approximately at correct velocity and vary smoothly with velocity, regardless of data bandwidth. We describe the first two cases explicitly; the third is similar. In all cases, the differential semblance function is

$$J[c,d] = \frac{1}{2}\int \mathrm{d}x \int \mathrm{d}z \int \mathrm{d}y\, |PV(x,z,y)|^2$$

where the bin parameter y and the DS operator P are as follows:

0. $y = h$ and $P = P_0$, $P_0V(x,z,h) = hV(x,z,h)$.

1. $y = p$ and $P = P_1$,

$$P_1V(x,z,p) = (2\pi)^{-1}\times \int\int u(x+h, x-h, \omega, z)\frac{1}{i\omega}\frac{\partial}{\partial p}\mathrm{e}^{-i\omega p\cdot h}\phi(h)\,\mathrm{d}h\,\mathrm{d}\omega. \tag{4}$$

$$= \int\int hu(x+h, x-h, \omega, z)\mathrm{e}^{-i\omega p\cdot h}\phi(h)\,\mathrm{d}h\,\mathrm{d}\omega.$$

The last equality reveals the close relation between the two DS operators P_0 and P_1. This relationship justifies calling the operator P_0 a differential semblance operator, even though it expresses concentration or focusing rather than invariance with respect to a bin parameter: it is actually the (x,t) representation of a bona fide differential semblance measure in (p,τ).

Numerical solution and gradient computation

Equation (1) is solved numerically by (i) approximating the DSR operator $F[c]$ using one of several methods which render its action computable with reasonable efficiency, and (ii) discretization of the approximation using spatial and/or spatial frequency gridding, coupled by discrete Fourier transform, and by an (explicit or implicit) depth-marching scheme.

An enormous literature exists on computable approximations to $F[c]$; see for example (Claerbout, 1985; De Hoop et al., 2000). (De Hoop et al., 2000) and references cited there describe so-called Generalized Screen propagators which approximate the square roots by sums of the form

$$\sum_{\alpha=1}^{L} f_\alpha(c(x,z),z,\omega)\,(\mathcal{F}_\text{x})^{-1}\,(g_\alpha(k,z,\omega)(\mathcal{F}_\text{x}u)(k,z,\omega))\,. \tag{5}$$

in which $\mathcal{F}_x$ denotes the Fourier transform in x. The series ceases to approximate the square root near $1 = c(x,z)k/\omega$ to suppress horizontally traveling energy.

Gridding an approximation as just described by discrete sampling and discrete Fourier transform produces a *discrete DSR operator* $\tilde{F}[c]$. Replacing $F[c]$ by $\tilde{F}[c]$ on the right-hand side of (1) leads to a system of ordinary differential equations. These can be solved approximately by an explicit or implicit method, the best choice depending on the characteristics of $\tilde{F}[c]$. We will represent implicit schemes as formally explicit. The result is a depth marching scheme of the form

$$u_{n+1} = u_n + \Delta z \Phi_n[c] u_n \tag{6}$$

in which $u_n = u_n(s,r,\omega) \simeq u(s,r,\omega,n\Delta z)$ is the downward continued field and $\Phi_n[c]$ is an extrapolation operator which derives from applying eg. a Runge-Kutta formula to $\tilde{F}[c]$. For the simplest scheme, forward Euler, $\Phi_n[c] = \tilde{F}[c(\cdot, n\Delta z)]$; more complex schemes involved combining values of $\tilde{F}[c(\cdot,z)]$ for various depths z. $\Phi_n[c]$ depends functionally on $c(x,z)$; the actual dependence depends on the choice of lateral and depth discretization, on the choice of approximation to the DSR operator, and on the choice of depth marching scheme.

After discretization, the differential semblance functions introduced in the last section have the form

$$J[c,d] = \frac{1}{2}\sum_n \sum_{x,y} |Pu_n|^2$$

in which P, the semblance operator, is independent of n (at least in our examples) and produces an output depending on midpoint x and a second variable y, which is offset, offset ray parameter, or vertical ray parameter according to the selection made in the last section.

The perturbation δJ in J resulting from a perturbation δc in c is

$$\delta J[c,d] = \sum_n \sum_{x,y} (Pu_n)(P\delta u_n)$$

$$= \sum_n \sum_{s,r} (P^* P u_n)(\delta u_n)$$

in which P^* denotes the *adjoint operator* to P and δu_n solves the linearization of (6):

$$\delta u_{n+1} = \delta u_n + \Delta z \Phi_n[c]\delta u_n + \Delta z \delta\Phi_n[c] u_n \tag{7}$$

Define the *adjoint field* $w_n(s,r,\omega)$ as the solution of the *adjoint state equation*

$$w_{n-1} = w_n + \Delta z \Phi[c]^* w_n + \Delta z P^* P u_n \tag{8}$$

which is to be solved in order of *descending* n (i.e. upward continuation) with initial condition $w_n = 0$ for large n, and the superscript $*$ denotes the *adjoint* or transpose operator. Then

$$\delta J[c,d] = \frac{1}{\Delta z}\sum_n \sum_{s,r} (w_{n-1} - w_n - \Delta z \Phi[c]^* w_n)\delta u_n$$

$$= \frac{1}{\Delta z}\sum_n \sum_{s,r} (w_n)(\delta u_{n+1} - \delta u_n - \Delta z \Phi[c]\delta u_n)$$

which is, from (7),

$$= \sum_n \sum_{s,r} w_n \delta\Phi_n[c] u_n$$

Now $\delta\Phi[c]u = \partial_c(\Phi[c]u)\delta c$, in which ∂_c denotes partial derivative with respect to c. Denote by $\Psi[c,u]$ the adjoint of the operator $\delta c \mapsto \partial_c(\Phi[c]u)\delta c$. Then

$$\delta J[c,d] = \sum_{x,z} (\sum_n \Psi[c,u_n] w_n)\delta c$$

from which we read off that the gradient of J is

$$\nabla_c J[c,d] = \sum_n \Psi[c,u_n] w_n$$

This method for computing the gradient of a function depending on the solution of a differential equation goes by the name *adjoint state method*, and originates in control theory (Lions, 1971). It was introduced into the theory of inverse problems by (Chavent and Lemonnier, 1974), and further developed by (Tarantola, 1987) amongst others.

To summarize, to compute the gradient:

1. Downward continue the wavefield (compute $u_n, n = 0,...,N$ by solving (6));

2. Upward continue the adjoint wavefield (compute $w_n, n = N,...0$ by solving (8));

3. Apply the operator $\Psi[c, u_n]$ to w_n, which amounts to a weighted crosscorrelation of u_n and w_n, and sum on n to obtain the gradient.

This computation involves several operators not ordinarily needed in wavefield extrapolation imaging:

1. P^* and P^*P (adjoint equation (8));

2. $\Phi[c]^*$ (adjoint equation (8));

3. $\Psi[c, u]$ (crosscorrelation operator for gradient synthesis).

The operator P_0 is self-adjoint, and the adjoints of P_1 and $\Phi_n[c]$ are straightforward to compute. Only $\Psi_n[c, u_n]$ might seem a bit mysterious, but in fact its computation is also quite straightforward. For example, for forward Euler depth stepping, $\Phi_n[c] = \tilde{F}[c(\cdot, n\Delta z)]$, so (ignoring discretization)

$$(\Psi_n[c, u_n]w_n)(x) = \mathrm{Re} \int \mathrm{d}\omega \sum_{\alpha=1}^{L} \frac{\partial f_\alpha}{\partial c}(c(x, n\Delta z), n\Delta z, \omega)$$

$$\times \left\{ \int dr \left[\mathcal{F}_{\mathrm{s}}^{-1} \left(g_\alpha(\cdot, n\Delta z, \omega) \left(\mathcal{F}_{\mathrm{s}} u_n\right)(\cdot, r, \omega)\right) \overline{w_n}\right](x, r, \omega)\right.$$

$$\left. + \int ds \left[\mathcal{F}_{\mathrm{r}}^{-1} \left(g_\alpha(\cdot, n\Delta z, \omega) \left(\mathcal{F}_{\mathrm{r}} u_n\right)(s, \cdot, \omega)\right) \overline{w_n}\right](s, x, \omega)\right\}$$

which is to be summed into $\nabla_c J[c, d](x, z)$ at $z = n\Delta z$.

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